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Use of quadratic differentials for description of defects and textures in liquid crystals and 2 + 1 gravity

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Abstract

The theory of measured foliations which is discussed in Part I in connection with the train tracks and meanders is shown to be related to the theory of Jenkins–Strebel quadratic differentials by Hubbard and Masur (Acta Math. 142 (1979) 221). In this work it is demonstrated that this formalism not only provides the adequate description of defects and textures in liquid crystals but also is idealy suited for study of 2 + 1 classical gravity which was initiated in the seminal paper by Deser et al. (Ann. Phys. 152 (1984) 220). Not only their results are reproduced but, in addition, many new results are obtained.In particular, using the results of Rivin (Ann. Math. 139 (1994) 553) the restriction on the total mass of the 2 + 1 Universe is removed. It is shown, that the masses can have only discrete values and, moreover, the theoretically obtained sum rules forbid the existence of some of these values. The dynamics of 2 + 1 gravity which is associated with the dynamics of train tracks (Part I), is reinterpreted in terms of the emerging hyperbolic 3-manifolds. The paper provides a concise introduction to this topic. The discussion of some connections of the obtained results with related physical problems is also provided. These include (but not limited to): string theory, classical and quantum billiards, dynamics of fracture, statics and dynamics of dislocations and disclinations in solids, etc.

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Keywords: Quadratic differentials; Defects in liquid crystals; Gravity

1. Introduction

In the previous paper [1] (Part I) we have discussed some aspects of the theory of measured foliations for 2-dimensional surfaces without providing sufficient mathematical proofs. In this part we would like to provide a self-contained mathematical justification of the

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obtained results.Our presentation is deliberately pedagogical enough to be accessible not only to the experts on gravity but to the interested readers in the areas of condensed matter physics.

The theory of measured foliations is mathematically connected with the theory of quadratic differentials as was demonstrated by Hubbard and Masur [2]. Therefore, we mainly will concentrate our efforts on this connection. The usefulness of the measured foliations for the description of defects and textures in liquid crystals was recognized for some time [3,4]. More recently,the usefulness of quadratic differentials for description of bosonic strings was recognized in Ref. [5]. Additional nontrivial results for strings which involve quadratic differentials were obtained in Ref. [6]. Here we argue that the quadratic differentials not only naturally occur in $2 + 1$ gravity but, in Section 7, we indicate that they may also be helpful in the theories of motion in classical and quantum billiards, theories of elasticity and dynamics of fracture, etc. In Section 2 we provide an auxiliary introduction to the isoperimetric inequalities. Although the existing monographs on quadratic differentials do not explicitly contain this information [7,8], nevertheless, they assume that the reader is familiar with it. We believe, that the isoperimetric inequalities provide the most natural background necessary for understanding of the physical meaning of quadratic differentials. Although most of the results presented in this section are known, they are scattered in the literature and this circumstance, we feel, justifies their presentation in this paper. In Section 3 these results acquire new meaning when we discuss essentials of quadratic differentials. In the mathematical physics literature there is already a good treatment of this topic, e.g. see Ref. [5], nevertheless, we feel, that our presentation could be considered as complementary. Our discussion in this section is subordinated to our intention to use the quadratic differentials in the theory of liquid crystals and gravity. Applications to liquid crystals are discussed in Section 4. The results of this section are aimed at explaining the physical meaning of quadratic differentials in terms of known (to physicists) results from differential geometry of surfaces with singularities. We also provide some proofs in support of the results discussed in Section 5 (Part I). In Section 5 (Part I) the surface energy, Eq. (5.3), was used in calculations without mathematical justification. Next, in Section 5 of *this* part of our work we discuss applications of quadratic differentials to $2 + 1$ gravity. The presentation of this section is strongly influenced by the seminal work of Deser et al. [9]. We reanalyze and extend their results using known in mathematical literature connection between the quadratic differentials and conical (surface) singularities [10]. We reobtain this connection in a way somewhat different from that given in Ref. [10] and then, we extend this connection with help of recently obtained more comprehensive mathematical results by Rivin [11]. This allows us to remove the restriction on the maximal total mass of $2 + 1$ Universe which follows from the results of Ref. [9]. Using the results of Hopf [12] we argue that the masses in such Universe should take only discrete values. That is, already at the classical level, we obtain a sort of quantization condition for masses. The results of this section are elaborated in Section 6 where more advanced topics are briefly discussed. In particular, we argue that not only masses should be quantized but, in addition, the obtained mass spectrum is subject to some selection rules which forbid existence of particles with certain values of quantized masses. This result follows directly from the theory of quadratic differentials [13]. We also briefly study the effects of inclusion of the cosmological term into Einstein's equations of 2 + 1 gravity at the classical level from both mathematical and physical points of view. This leads us to the connection with the nonperturbative treatment of bosonic string theory developed by Takhtadjian [14]. While in Part I (Section 4) we had developed theory of measured foliations based on the train tracks here, we provide (in Section 6) connections of this theory and the theory of Teichmüller spaces in order to clarify the physical processes responsible for the topology nonpreserving moves for train tracks which are depicted in Fig. 24 (Part I). These processes are responsible for phase changes (reducible, periodic, pseudo-Anosov) discussed in Part I. Finaly, we briefly discuss how the motion in Teichmüller space is related to the evolution in the Minkovski spacetime. This treatment is complementary to that developed by Moncrief [15] and serves to sketch the connections between 2 + 1 gravity, hyperbolic 3-manifolds and the theory of knots and links (for some additional gravity-nonrelated physical applications of the theory of knots and links, please, consult Ref. [16]). Some important auxiliary results related to knots/links and the hyperbolic 3-manifolds are presented in Appendix A. This is motivated by the fact, that from the mathematical standpoint, the existence of knots and links in 2 + 1 gravity is by no means self-obvious. In the main text we provide sufficient arguments in order to demonstrate the existing difficulties. In Appendix A we provide some sketch of very recent mathematical results in support of the existence of knots and links in 2 + 1 gravity. Our exposition of mathematical results related to hyperbolic 3-manifolds is rather terse (unlike other subjects which we treat with sufficient details). Nevertheless, it is included in this paper since it is logically connected with the rest of it. We plan to return to these more advanced subjects in future publications.

Related physical problems which could be treated by the methods developed in Parts I and II are briefly listed in Section 7. These problems include (but not restricted to): classical and quantum billiards, dynamics of fracture, dislocations in solids, Temperley–Lieb algebra related to meanders and its connection with the invariants of 3-manifolds, etc.

2. Some important isoperimetric inequalities

Consider a closed simple (without self-intersections) planar curve C of length L and let A be the area which is enclosed by C, then

$$L^2 \geq 4\pi A, \tag{2.1}$$

where the equality holds only for a circle. Physical applications of this inequality had been recently discussed in Ref. [17,103]. Here, we are mainly concerned with the methods of proving the inequality (2.1).

Since analytically the length L is given by

$$L = \int_a^b \mathrm{d}\tau \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2} \tag{2.2}$$

and the area A is known to be given by

$$A=-\int_a^b \mathrm{d}\tau y\frac{\mathrm{d}x}{\mathrm{d}\tau}, \tag{2.3}$$

we can use a simple identity (obtained with help of parametrization: $\tau=(2\pi/L)s$)

$$\int_0^{2\pi}\mathrm{d}\tau\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2\right]=\int_0^{2\pi}\mathrm{d}\tau\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2=\frac{\mathrm{L}^2}{2\pi} \tag{2.4}$$

in order to obtain

$$\begin{aligned}L^2-4\pi A&=2\pi\int_0^{2\pi}\mathrm{d}\tau\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2+2y\frac{\mathrm{d}x}{\mathrm{d}\tau}\right]\\&=2\pi\int_0^{2\pi}\mathrm{d}\tau\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)+y\right)^2+2\pi\int_0^{2\pi}\mathrm{d}\tau\left[\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2-y^2\right].\end{aligned} \tag{2.5}$$

Hence, to prove the inequality (2.1) we have to prove that

$$\int_0^{2\pi}\mathrm{d}\tau\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2\geq\int_0^{2\pi}\mathrm{d}\tau y^2. \tag{2.6}$$

This, however, may or *may not* be the case, e.g. if $y=\text{const}$ the above inequality certainly fails. It is rather easy to prove [18] the following theorem.

Theorem 2.1. *If $y(\tau)$ is a smooth function with period 2π and if $\int_0^{2\pi}\mathrm{d}\tau y(\tau)=0$, then the inequality* (2.6) *holds and becomes an equality if and only if $y(\tau)=a\cos\tau+b\sin\tau$, i.e. when the trajectory is a circle.*

The above result (2.1) can be extended for the case when the curve may have self-intersections [19]. We are not going to need this case, however. Instead, we shall consider the extension of this result to the nonflat surfaces. In the case of a sphere of radius R, it could be shown [20] that

$$L^2\geq 4\pi A-\frac{A^2}{\mathrm{R}^2} \tag{2.7}$$

with equality holding only for a circle on the sphere. In the case of a pseudosphere (surface of constant negative curvature) of radius $R=\mathrm{i}$, $\mathrm{i}=\sqrt{-1}$, we obtain, instead of (2.7), the following result:

$$\frac{L^2}{A}\geq A+4\pi. \tag{2.8}$$

Obviously, if (2.8) holds, then the inequality of the type given by (2.1) holds as well. Beacause of this, following Ahlfors [21], let us define the extremal length $\lambda(\Gamma)$ via

$$\lambda(\Gamma)=\sup_\rho\frac{L^2(\rho)}{A(\rho)}, \tag{2.9}$$

where Γ is some closed set of curves and ρ is the metric of the surface defined in such a way that

$$L_C(\rho) = \int_C \rho |\,\mathrm{d}z| \tag{2.10}$$

and

$$A_\Delta(\rho) = \iint_\Delta \rho^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{2.11}$$

where Δ is the area enclosed by the contour $C \in \Gamma$ and $z = x + \mathrm{i}y$.

Remark 2.2. *The above definition of $\lambda(\Gamma)$ can be extended to open curves as well if we have surfaces with boundaries.*

To get a feeling of the above results, let us consider a rectangle $\hat{R}$ with sides a and b and let C be some curve which joins the opposite sides of the rectangle. Then, for any ρ

$$\int_0^a \rho(x + \mathrm{i}y)\,\mathrm{d}x \geq L_C(\rho) \tag{2.12}$$

and, from here,

$$\int_0^b \int_0^a \rho \,\mathrm{d}x\,\mathrm{d}y \geq bL_C(\rho). \tag{2.13}$$

With the help of (2.11) and (2.13) we obtain as well

$$b^2 L_C^2(\rho) \leq ab \int_0^a \int_0^b \rho^2 \,\mathrm{d}x\,\mathrm{d}y = abA_{\hat{R}}(\rho). \tag{2.14}$$

The above result was obtained with help of the Schwarz-type inequality [22]

$$\left[\iint_{\hat{R}} \rho \cdot 1 \,\mathrm{d}x\,\mathrm{d}y\right]^2 \leq \iint_{\hat{R}} \rho^2 \,\mathrm{d}x\,\mathrm{d}y \iint_{\hat{R}} \mathrm{d}x\,\mathrm{d}y \cdot 1 = ab \iint_{\hat{R}} \rho^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{2.15}$$

The inequality (2.14) produces, in turn,

$$\frac{a}{b} \geq \frac{L_C^2(\rho)}{A_{\hat{R}}(\rho)} \equiv \lambda(\Gamma). \tag{2.16}$$

At the same time, if we choose $\rho = 1$ inside $\hat{R}$ (and $\rho = 0$ outside), then $L_C(\rho) = a$ and $ab = A(\rho)$ so that $\lambda(\Gamma) \geq a/b$. From here we arrive at the conclusion, that for the rectangle $\hat{R}$

$$\frac{a}{b} = \lambda(\hat{R}). \tag{2.17}$$

By definition, the *modulus* $M = b/a = \lambda^{-1}$ or, in general,

$$M(\Gamma) = \inf_\rho \frac{A(\rho)}{L_C^2(\rho)}. \tag{2.18}$$

With help of M we obtain for the rectangle $\hat{R}$ the following result:

$$A_{\hat{R}}(\rho) = \iint_{\hat{R}} \rho^2 \,\mathrm{d}x\,\mathrm{d}y \geq L_C^2 M \geq a^2 M = ab. \tag{2.19}$$

This happens to be the central result for the entire development as we shall demonstrate below in the rest of this work. To this purpose, let us discuss the related problem about calculation of the modulus M for the annulus $\check{D}$, i.e. doubly connected region made of two concentric rings C_1 and C_2 of radius $r_1 < r_2$. By choosing the polar system of coordinates, we obtain, by analogy with (2.12),

$$L_C(\rho) \leq \int_0^{2\pi} \rho(r\mathrm{e}^{\mathrm{i}\varphi}) r \,\mathrm{d}\varphi \tag{2.20}$$

for any closed curve *which separates* C_1 *and* C_2. From here,

$$\frac{L_C(\rho)}{r} \leq \int_0^{2\pi} \rho \,\mathrm{d}\varphi$$

and

$$L_C(\rho) \ln\left(\frac{r_2}{r_1}\right) \leq \iint_{\check{D}} \rho \,\mathrm{d}r\,\mathrm{d}\varphi. \tag{2.21}$$

This resembles very much (2.13) and, therefore, by analogy with (2.14) we obtain,

$$L_C^2(\rho) \ln^2\left(\frac{r_2}{r_1}\right) \leq \left[\iint_{\check{D}} \rho \,\mathrm{d}r\,\mathrm{d}\varphi\right]^2. \tag{2.22}$$

Again, using the Schwarz inequality,

$$\left[\iint_{\check{D}} \frac{1}{r}\rho r \,\mathrm{d}r\,\mathrm{d}\varphi\right]^2 \leq \left[\int_{r_1}^{r_2} \mathrm{d}r \int_0^{2\pi} \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right] \cdot \left[\iint_{\check{D}} \rho r \,\mathrm{d}r\,\mathrm{d}\varphi\right], \tag{2.23}$$

we finally obtain,

$$L_C^2(\rho) \ln^2\left(\frac{r_2}{r_1}\right) \leq 2\pi \ln\left(\frac{r_2}{r_1}\right) \iint_{\check{D}} r\rho \,\mathrm{d}r\,\mathrm{d}\varphi. \tag{2.24}$$

This produces immediately

$$\frac{L^2(\rho)}{A(\rho)} \leq \frac{2\pi}{\ln(r_2/r_1)}. \tag{2.25}$$

If we choose $\rho = a/2\pi r$ (where a is length parameter which is determined in Eq. (2.31)), then (2.25) is converted into equality with $L(\rho) = a$ and $A(\rho) = (a^2/2\pi)\ln(r_2/r_1)$. As in the case of a rectangle, we conclude, that

$$M(\check{D}) = \frac{1}{2\pi} \ln\left(\frac{r_2}{r_1}\right). \tag{2.26}$$

In accord with (2.19), we can rewrite inequality (2.25) as

$$A(\check{D}) \geq L_C^2(\rho) M(\check{D}) \geq a^2 M. \tag{2.27}$$

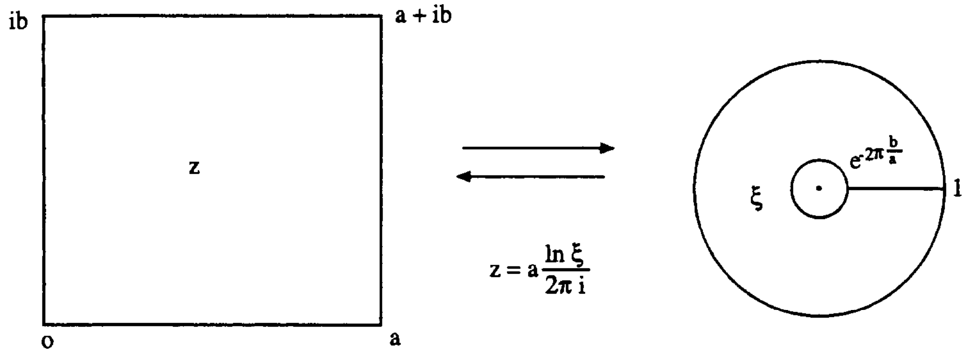

Fig. 1. Conformal mapping between the rectangle and the annulus.

Let us observe that, actually, the rectangle $\hat{R}$ and the annulus $\check{D}$ could be mapped conformally into each other. Moreover, it can be shown [21] that $\lambda(\Gamma)$ is conformal invariant. This can be easily understood if we notice that for any conformal mapping $z \to \tilde{z}$ we should have, by construction,

$$\rho|\,\mathrm{d}z| = \tilde{\rho}|\,\mathrm{d}\tilde{z}| \tag{2.28}$$

and, accordingly, for the area $\mathrm{d}A$

$$\mathrm{d}A = \rho^2\,\mathrm{d}x \wedge \mathrm{d}y = \rho^2 \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = \tilde{\rho}^2 \frac{\mathrm{i}}{2}\,\mathrm{d}\tilde{z} \wedge \mathrm{d}\bar{\tilde{z}}. \tag{2.29}$$

The invariance of $\mathrm{d}A$ follows directly from the invariance of the length element $\mathrm{d}l = \rho|\,\mathrm{d}z|$ as can be checked directly from Eq. (2.28) [5]. Clearly, the metric ρ determines all surface properties. In particular, the singularities of the metric correspond to some surface singularities (defects) as we shall demonstrate.

In the meantime, let us consider in some detail the conformal mapping of the rectangle $\hat{R}$ onto $\check{D}$ .To facilitate our understanding, it is useful to visualize the mapping, e.g. see Fig. 1. The mapping from ξ-plane ($\check{D}$) to z-plane ($\hat{R}$) is being performed by the following equation:

$$z = a\frac{\ln \xi}{2\mathrm{i}\pi}. \tag{2.30}$$

It maps the annulus which is cut along the positive ξ-axis into the rectangle. The cut could be avoided if we convert the rectangle $\hat{R}$ into a cylinder of height b (by identifying $[0, \mathrm{i}b]$ with $[a, a + \mathrm{i}b]$). The periodicity is most explicitly seen by rewriting Eq. (2.30) in the form

$$\xi = \exp\left(\frac{2\mathrm{i}\pi z}{a}\right) \tag{2.31}$$

so that, obviously, $\xi(z) = \xi(z + a)$. Taking into account Eq. (2.26) and using Eq. (2.30) we obtain,

$$M(\check{D}) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right), \tag{2.32}$$

where $(1/2\pi)\ln(1/r) = b/a$. Using Eq. (2.27) we obtain,

$$A(\check{D}) \geq a^2 M = ab, \tag{2.33}$$

which coincides *exactly* with the result (2.19) as anticipated. Finally, for an annulus $\check{D}$ in which some curve C connects the inner circle C_1 with the outer circle C_2 simple calculation shows [21]:

$$\frac{L_C^2(\rho)}{A(\rho)} \leq \frac{1}{2\pi} \ln\left(\frac{r_2}{r_1}\right). \tag{2.34}$$

Compare this with Eq. (2.25). If, as before, we choose $r_2 = 1$ and $r_1 = r$ then, by definition,

$$\hat{M} = \frac{2\pi}{\ln(1/r)} = \frac{a}{b}, \tag{2.35}$$

so that $A(\check{D}) \geq b^2 \hat{M} = ab$.

The duality between the results (2.34) and (2.25) happens to be very important as we shall demonstrate in Section 4. In the meantime, we need to introduce the concept of a quadratic differential. This is accomplished in the next section.

3. Some essentials about quadratic differentials

Let us begin with Eq. (2.31).Using this equation we obtain the following sequence of results:

$$\mathrm{d}\xi = \frac{2\mathrm{i}\pi}{a}\xi\,\mathrm{d}z \longrightarrow \mathrm{d}z = \frac{a}{2\mathrm{i}\pi}\frac{\mathrm{d}\xi}{\xi} \longrightarrow (\mathrm{d}z)^2 = -\frac{a^2}{(2\pi)^2}\frac{(\mathrm{d}\xi)^2}{\xi^2}. \tag{3.1}$$

The last expression represents the first example of a quadratic differential. More formally, we provide the following definition.

Definition 3.1. Let $z = f(\tilde{z})$ be some coformal mapping of the domain $\tilde{D}$ onto D and let $\varphi(z)$ be some function wich transforms as

$$\varphi(\tilde{z}) = \varphi(z)\left(\frac{\mathrm{d}z}{\mathrm{d}\tilde{z}}\right)^2, \quad z = f(\tilde{z}), \tag{3.2}$$

then the expression $\varphi(z)(\mathrm{d}z)^2$ is called quadratic differential.

From (3.2) it follows, that this quantity is invariant with respect to mappings: $z \to \tilde{z}$, i.e.

$$\varphi(z)(\mathrm{d}z)^2 = \varphi(\tilde{z})(\mathrm{d}\tilde{z})^2. \tag{3.3}$$

Eq. (3.1) represents just an example of general result given by Eq. (3.3). Comparison between Eqs. (2.28) and (3.3) suggests us to introduce the following definition.

Definition 3.2. The differential $|\mathrm{d}w| = |\varphi(z)|^{1/2}|\mathrm{d}z|$ is called the length element (φ-length) associated with φ.

Remark 3.3. (a) *In terms of* $|\mathrm{d}w|$ *the length is just the usual Euclidean length;* (b) $\rho(z)$ *can be identified, in principle, with* $|\varphi(z)|^{1/2}$ *but this association is actually formal as we shall explain in Section* 4 .

From the above discussion it follows, in particular, that

$$1 \cdot (\mathrm{d}w)^2 = \varphi(z)(\mathrm{d}z)^2. \tag{3.4}$$

Suppose, that there is another $\tilde{w}$ such that

$$1 \cdot (\mathrm{d}\tilde{w})^2 = \varphi(z)(\mathrm{d}z)^2. \tag{3.5}$$

Then, clearly, $\tilde{w} = \pm w + \text{const}$. Moreover,by taking a square root of Eq. (3.4) we obtain,

$$w = \Phi(z) = \int^z \mathrm{d}z\sqrt{\varphi(z)}. \tag{3.6}$$

From here, we obtain as well

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \sqrt{\varphi(z)}. \tag{3.7}$$

This is the differential equation on the complex plane or on the Riemann surface, etc. The flow lines of this equation were used extensively in Part I of this work. Now, we would like to provide some additional details. To this purpose, let us consider quadratic differential of the type

$$(\mathrm{d}w)^2 = \left(\frac{n+2}{2}\right)^2 z^n(\mathrm{d}z)^2, \tag{3.8}$$

where n is some integer ($n \geq -1$). Use of Eq. (3.6) provides us with the result,

$$w = \Phi(z) = z^{(n+2)/2}, \tag{3.9}$$

or

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \left(\frac{n+2}{2}\right) z^{n/2}. \tag{3.10}$$

Eq. (3.9) is a typical example of a conformal mapping. For such mapping, with respect to the origin of z-plane, the whole z-plane is subdivided into sectors ($n \geq -1$)

$$\frac{2\pi}{n+2}k \leq \arg z \leq \frac{2\pi}{n+2}(k+1), \quad k = 0, 1, \ldots, n+1, \tag{3.11}$$

such that when arg z covers one of these sectors,the resulting image covers either the upper or the lower w-plane. If we consider the set of parallel lines(parallel to Re w axis in w-plane), Im $w = an$, $a = \text{const}$, $n = 0, 1, 2, \ldots,$ these parallel lines go into "parallel lines" in one of the sectors of z-plane so that for the Y-type singularity we obtain the result depicted in Fig. 2. In the case of a thorn, $n = -1$, and we have the mapping of the whole z-plane (with cut along $x \geq 0$ axis) into the upper w-plane. The horizontals in w-plane

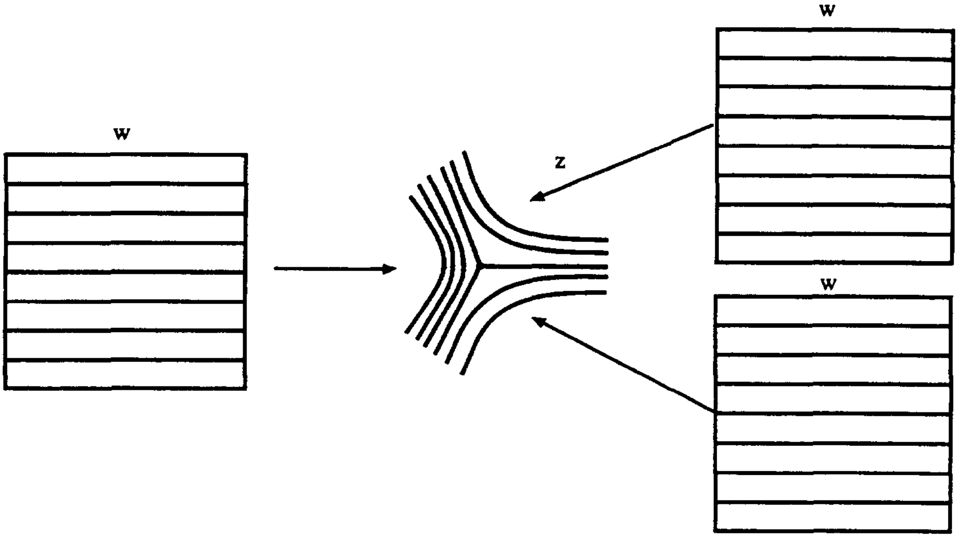

Fig. 2. Conformal mapping between w and z planes which explains how Y-type singularity can be built out of "flat bricks"(rubber bands).

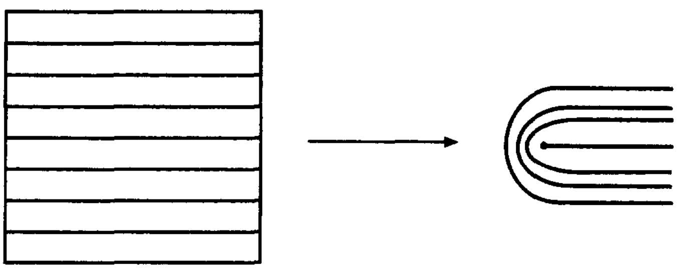

Fig. 3. Conformal mapping between w and z planes which explains how thorn can be built from the "flat brick" (rubber band).

go into the "horizontals" in z-plane as depicted in Fig. 3. Straightening of the flow lines explains the reason of the word "measured foliations" introduced in Part I (e.g. see Fig. 21 of Part I). From the examples of the previous section it follows, that the singularities producing poles of order > 2 are not acceptable since if we identify $|\varphi(z)|$ with $\rho^2(z)$, then the area, Eq. (2.11), becomes divergent. This means, in particular, that the singularities depicted in Fig. 4 and known in the literature on liquid crystals [23] are mathematicaly ill-defined (see also Section 4 for additional details). The singularities for which $n > 1$ are permissible, in principle, and could be considered along the lines similar to that discussed in Part I (see Ref. [24] and Section 6).

All mathematically permissible quadratic differentials can be brought into the *standard form* as follows. Consider the mapping

$$\xi(z) = \exp\left(\frac{2i\pi}{L_\Phi}\Phi(z)\right). \tag{3.12}$$

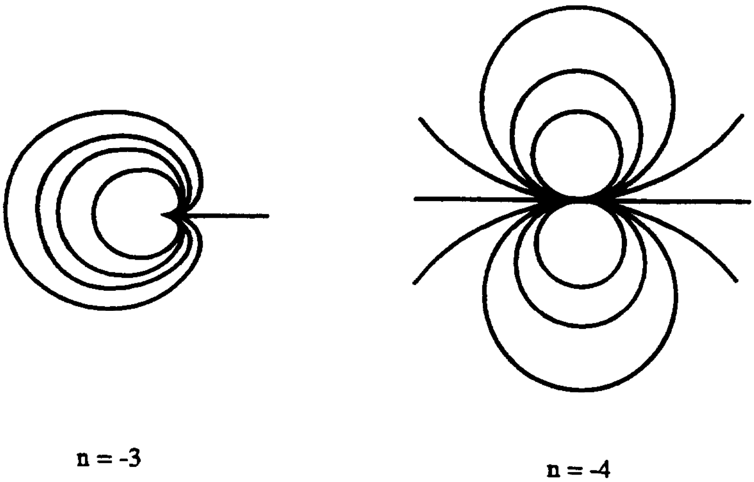

Fig. 4. Some of the "forbidden" sigularities.

It is analogous to Eq. (2.31) and L_Φ is φ-length of the quadratic differential $\Phi(z)$. By analogy with Eq. (3.1), we have now

$$\mathrm{d}\xi = \frac{2\mathrm{i}\pi}{L_\Phi}\xi\sqrt{\varphi(z)}\,\mathrm{d}z \leftrightarrows \varphi(z)(\,\mathrm{d}z)^2 = -\frac{L_\Phi^2}{(2\pi)^2}\frac{(\,\mathrm{d}\xi)^2}{\xi^2} \tag{3.13}$$

The flow lines of such defined quadratic differential are the *concentric circles* (Fig. 4, Part I). This is a very important result since it allows to map an arbitrary mathematicaly permissible quadratic differential into the standard differential equation (3.1) (or (3.13)), so that the rest of the arguments related to the annulus and to the rectangle presented in Section 2 could be carried through without change.

The above results can be broadly generalized now. To this purpose let us take another look at Eq. (3.12). What we actually have is a mapping from some Riemann surface R on which the quadratic differential "lives" into the surface of the flat annulus. Clearly, such mapping has some limitations. That is, it might as well be that the above mapping exist only for some ring domain $|z| = \rho$ on R. The size of this domain determines the size of the annulus (or punctured disk). Without going into intricate details about the correspondence of these domains [25,26] we provide the following theorem.

Theorem 3.4. *If φ is quadratic differential which can have closed trajectories on R with respective lengths L_{φ_i}, then its characteristic ring domains (i.e. the maximal ring domains swept out by the closed trajectories) cover R up to a set of measure zero. For a holomorphic quadratic differential on a compact surface of genus $g \geq 2$ the number of characteristic ring domains is at most $3g - 3$.*

Proof. Please, consult Refs. [7,8,25,26]. □

Let us explain what all this actually means. It is well known [27], that every Riemann surface R could be decomposed into set of $2g - 2$ pants. Conversely,if we have at our

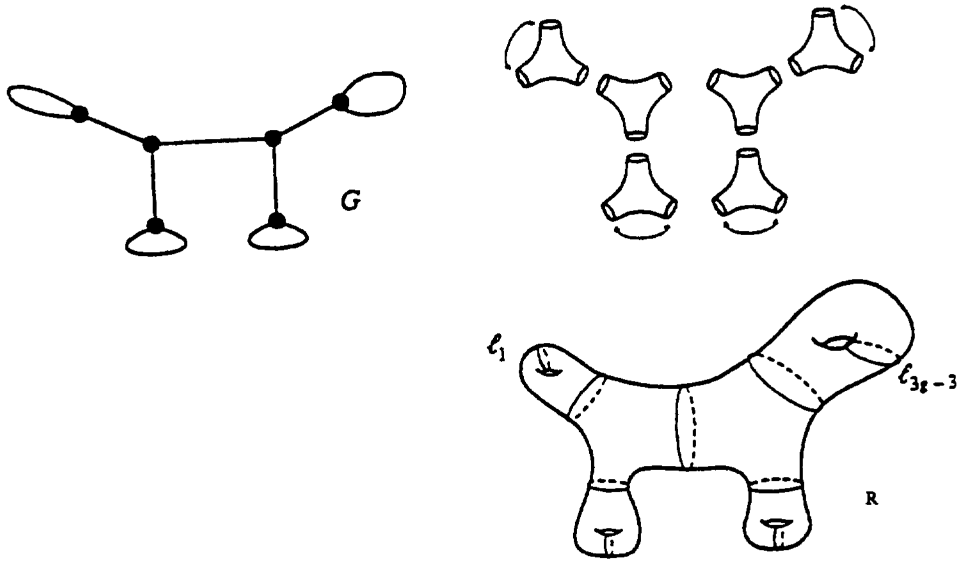

Fig. 5. The topological structure of every closed Riemann surface R is encoded in 3-valent planar graph G. With the help of this graph the pants decomposition can be found. The graph G provides the gluing prescription for such pants decomposition.

disposal a set of $2g-2$ pants, we can restore R if we have a 3-valent planar graph with $2g-2$ vertices and $3g-3$ edges which does not have free ends. $3g-3$ edges correspond to cylinders made when the pants are glued together. All this is depicted in Fig. 5. Since we have discussed already the problems which involve cylinders, e.g. see Fig. 1, it becomes clear why we have discussed them in the first place. Moreover, the union of closed trajectories in all ring domains effectively forms the union of closed geodesics (lamination, according to Definition 4.1. of Part I), one for each of these cylinders [27-29]. And, when these geodesics are projected into an open disk D^2 (as discussed in Section 4 of Part I), we obtain the projective meanders.

With this background, we are ready now for some applications.

4. Applications of quadratic differentials to textures in liquid crystals

In the previous sections we have provided essential mathematical background related to quadratic differentials. Before we actualy use them (in this and the following sections) it is desirable to provide some relevant physical background related to quadratic sdifferentials.

4.1. Differential geometric meaning of quadratic differentials

We have introduced the length, Eq. (2.28),and the area, Eq. (2.29), and identified $|\varphi(z)|^{1/2}$ with ρ and $|\varphi(z)|$ with ρ^2. This, however, is not enough. Here we shall explain why.

Although the Cartan method of description of surfaces is the most elegant and economical, moreover, it is indispensable for the discussion of general topological properties of surfaces [30], we shall avoid its use here for the reasons which will become obvious upon reading.

Let $\mathbf{r}$ be the point in R^3 which belongs to some surface S. Then, on one hand, $\mathbf{r}$ is just some 3d Euclidean vector with components r_1, r_2, and r_3 and, on the other hand, because it belongs to the surface, its location on the surface could be given in terms of some local coordinates $x_1, x_2 \in S$. With such defined $\mathbf{r}$ it is useful to construct two vectors $\mathrm{d}\mathbf{r}/\mathrm{d}x_1$ and $\mathrm{d}\mathbf{r}/\mathrm{d}x_2$ so that the induced metric of the surface is given by

$$\mathrm{d}l^2 = E\,\mathrm{d}x_1^2 + 2F\,\mathrm{d}x_1\,\mathrm{d}x_2 + G\,\mathrm{d}x_2^2, \tag{4.1}$$

where $E = (\mathrm{d}\mathbf{r}/\mathrm{d}x_1)^2$, $F = (\mathrm{d}\mathbf{r}/\mathrm{d}x_1)\cdot(\mathrm{d}\mathbf{r}/\mathrm{d}x_2)$, $G = (\mathrm{d}\mathbf{r}/\mathrm{d}x_2)^2$. Introduce now the unit normal vector to the surface $\mathbf{T}$ via

$$\boldsymbol{T} = \frac{(\mathrm{d}\mathbf{r}/\mathrm{d}x_1)\wedge(\mathrm{d}\mathbf{r}/\mathrm{d}x_2)}{|(\mathrm{d}\mathbf{r}/\mathrm{d}x_1)\wedge(\mathrm{d}\mathbf{r}/\mathrm{d}x_2|}, \tag{4.2}$$

then the second fundamental form of surface can be written as

$$\mathrm{II} = L\,\mathrm{d}x_1^2 + 2M\,\mathrm{d}x_1\,\mathrm{d}x_2 + N\,\mathrm{d}x_2^2, \tag{4.3}$$

where $L = -(\mathrm{d}\mathbf{T}/\mathrm{d}x_1)\cdot(\mathrm{d}\mathbf{T}/\mathrm{d}x_1)$, $M = -(\mathrm{d}\mathbf{T}/\mathrm{d}x_1)\cdot(\mathrm{d}\mathbf{r}/\mathrm{d}x_2)$ and $N = -(\mathrm{d}\mathbf{T}/\mathrm{d}x_2)\cdot(\mathrm{d}\mathbf{r}/\mathrm{d}x_2)$.

In terms of the quantities just defined one can determine the Gauss (intrinsic) curvature K as

$$K = k_1 k_2 = \frac{LN - M^2}{\hat{\lambda}^2} \tag{4.4}$$

and the mean (extrinsic) curvature H as

$$H = \frac{1}{2}(k_1 + k_2) = \frac{L + N}{2\hat{\lambda}}, \tag{4.5}$$

where the factor $\hat{\lambda}$ is associated with the first fundamental form which is brought into the conformal form

$$\mathrm{d}l^2 = \hat{\lambda}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \tag{4.6}$$

and k_1 and k_2 are the principal curvatures of the surface.

The displacement of the vector $\mathbf{r}$ along some curve which belongs to S is determined by the vector $\mathbf{t}$, i.e.

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = \frac{\mathrm{d}x_1}{\mathrm{d}l}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x_1} + \frac{\mathrm{d}x_2}{\mathrm{d}l}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x_2}. \tag{4.7}$$

According to the rules of differential geometry of curves [12], we have

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}l} = \kappa\mathbf{T}, \tag{4.8}$$

where κ is the local curvature of the *curve* which belongs to S. Depending upon how the line is drawn on the surface, κ may vary from k_1 to k_2 (if $k_1 > k_2$). The lines along which $\kappa = k_1$ (or k_2) are called the principal curvature directions.They form an orthogonal network.The equation for the lines of principal curvatures is given by

$$-M\,\mathrm{d}x_1^2 + (L-N)\,\mathrm{d}x_1\,\mathrm{d}x_2 + M\,\mathrm{d}x_2^2 = 0. \tag{4.9}$$

Finally, the Codazzi equations (the consistency equations) could be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\frac{L-N}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}x_2}M = \hat{\lambda}\frac{\mathrm{d}}{\mathrm{d}x_1}H, \tag{4.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_2}\left(\frac{L-N}{2}\right) - \frac{\mathrm{d}}{\mathrm{d}x_1}M = -\hat{\lambda}\frac{\mathrm{d}}{\mathrm{d}x_2}H. \tag{4.10b}$$

Introduce now the complex notations: $z = x_1 + \mathrm{i}x_2$ and take into account that for an arbitrary complex function $F(z, \bar{z}) = F_1 + \mathrm{i}F_2$ one can write

$$2\frac{\mathrm{d}}{\mathrm{d}z}F = \left(\frac{\mathrm{d}}{\mathrm{d}x_1}F_1 + \frac{\mathrm{d}}{\mathrm{d}x_2}F_2\right) - i\left(\frac{\mathrm{d}}{\mathrm{d}x_2}F_1 - \frac{\mathrm{d}}{\mathrm{d}x_1}F_2\right)$$

and another equation which is the complex conjugate of this. Then, following Hopf [4], we can introduce the Hopf differential

$$\Phi(z, \bar{z}) = \frac{L-N}{2} - \mathrm{i}M. \tag{4.11}$$

It could be shown, that

$$\frac{|\Phi|}{\lambda} = \left|\frac{k_1 - k_2}{2}\right|. \tag{4.12}$$

That is *the umbilic points* ($k_1 = k_2$) *are zeros of* Φ. With help of Φ both of the Codazzi equations could be rewritten in the compact form given by

$$\frac{\partial \Phi}{\partial \bar{z}} = \hat{\lambda}\frac{\partial}{\partial z}H. \tag{4.13}$$

Moreover, Φ itself can be also rewritten in more compact form as

$$\Phi = -2\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}z}\cdot\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}z}. \tag{4.14}$$

Let us consider now the change of parametrization of the surface, i.e. $x_1, x_2 \longrightarrow x_1', x_2'$ and let $\xi = x_1' + \mathrm{i}x_2'$. Clearly, we expect

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}z} = \frac{\partial \mathbf{r}}{\partial \xi}\frac{\mathrm{d}\xi}{\mathrm{d}z} \tag{4.15a}$$

and

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}z} = \frac{\partial \mathbf{T}}{\partial \xi}\frac{\mathrm{d}\xi}{\mathrm{d}z}. \tag{4.15b}$$

The combined use of Eqs. (4.14) and (4.15) produces

$$\Phi(z, \bar{z})(\mathrm{d}z)^2 = \Phi(\xi, \bar{\xi})(\mathrm{d}\xi)^2 \tag{4.16}$$

which coincides with the transformation rule for the quadratic differentials (3.3). Hence Φ is the quadratic differential! Moreover, the equation for the principal curvatures (4.9), can now be rewritten in compact suggestive form as

$$\mathrm{Im}[\Phi(\mathrm{d}z)^2] = 0 \tag{4.17}$$

which is equivalent to

$$\arg\Phi + 2\arg(\mathrm{d}z) = m\pi, \quad m = 0, \pm 1, \ldots. \tag{4.18}$$

This result can be conveniently rewritten as

$$\arg(\mathrm{d}z) = \frac{m\pi}{2} - \frac{1}{2}\arg\Phi. \tag{4.19}$$

Let we have now the isolated umbilic ($k_1 = k_2$) point p on the surface. Let us surround this point by some closed non-self-intersecting contour C and let us define the index of such point $I(p)$ as

$$I(p) = \frac{1}{2\pi}\delta(\arg(\mathrm{d}z)), \tag{4.20}$$

where δ means the variation of an angle when one is circling around C counterclockwise. Combining of Eqs. (4.19) and (4.20) produces

$$I(p) = -\frac{1}{2\pi}\frac{1}{2}\delta(\arg\Phi). \tag{4.21}$$

Assume now that locally Φ can be written as

$$\Phi(z, \bar{z}) = cz^n + \cdots. \tag{4.22}$$

For the umbilic point $n > 0$ and $\hat{\lambda}$ is nonsingular for $k_1 \neq k_2$. In addition,for $k_1 \neq k_2$ we may have singular $\hat{\lambda}$ (then we may have $n < 0$). To calculate the index of such singularity explicitly, we use a known fact of complex analysis [31],

$$\delta(\arg\Phi) = \mathrm{Im}\oint_C \frac{\mathrm{d}\Phi}{\Phi}. \tag{4.23a}$$

This produces, in view of Eqs. (4.21) and (4.23),

$$I(p) = -\frac{n}{2}, \quad n = \pm 1, \pm 2, \ldots. \tag{4.23b}$$

Definition 4.1. We shall call Eq. (4.23a) the Hopf quantization rule.

This result produces exactly the indices known for the disclinations in liquid crystals, e.g. see Section 1 (Part I) or Ref. [23]. In the case of Y-type singularity, we have, according to Strebel [7], $n = +1$, which produces $I_Y = -\frac{1}{2}$ in accord with Fig. 2 of Part I. For the case of a thorn, we have $n = -1$, which produces $I_I = \frac{1}{2}$ again, in complete accord with

Fig. 2 of Part I and with Refs. [12,32]. The index for the rest of the singularities can be computed now analogously.

Remark 4.2. *Poincare* [33] *had originally considered singularities of differential equations of the form*

$$a(x, y)\,\mathrm{d}x + b(x, y)\,\mathrm{d}y = 0. \tag{4.24}$$

This is just the condition of orthogonality between the vector (a, b) and the curve $(\mathrm{d}x,\ \mathrm{d}y)$. By going around a closed curve C the direction of the vector (a, b) can change only by $2\pi n$ where n is an integer. Hence, by having half integer values for I *is equivalent of not having vector fields on the surfaces. This was actually stated in Part* I *(Section* 1*) without proof.*

4.2. *Quadratic differentials and the textures in liquid crystals*

The fact that the textures in liquid crystals can be associated with measured foliations was recognized already by Poenaru [3] and Langevin [4]. The fact that the measured foliations are asssociated with quadratic differentials was explained by Hubbard and Masur [2]. Comprehensive treatment of measured foliations could be found in Refs. [24,34,35]. Here, we only provide some details to make this identification complete. Before doing so, we would like to mention that the results below are equally applicable to any kind of "hyperbolic paper" (the terminology used by Thurston [36]) that is to any surface which may develop some crumples (see also Section 7).

The free energy of distortion $\mathcal{F}_\mathrm{d}$ in the case of nematics (in one constant approximation) is given by [37]

$$\mathcal{F}_\mathrm{d} = \frac{1}{2}\tilde{K}\int \mathrm{d}^3r(\nabla\mathbf{n})^2, \tag{4.24}$$

where the coupling constant $\tilde{K}$ has dimensionality energy/cm and is related effectively to the surface tension. The coupling constant $\tilde{K}$ exibits therefore noticeable temperature dependence. The unit vector $\mathbf{n} = \mathbf{n}(\mathbf{r})$ is known in the literature on liquid crystals as *director*. On one hand, it is being used to describe the orientation of the individual molecule (usually very stiff, rod-like, organic molecule of not too high molecular weight) with respect to some preassigned fixed axis, on another hand, $\mathbf{n}$ in Eq. (4.24) is also called a director although it is no longer directly associated with the individual molecule. It is possible, however, to obtain the macroscopic distortion energy $\mathcal{F}_\mathrm{d}$ from the underlying microscopic molecular model of nematic liquid crystal [38]. One is usually looking for a minimum of $\mathcal{F}_\mathrm{d}$ under the additional constraint: $\mathbf{n}^2 = 1$ (the nonlinear sigma model [39]). In mathematical literature [40,41], the same problem is stated somewhat more precisely. Specifically, let $\varphi(\mathbf{x} = \mathbf{x}/|\mathbf{x}| = \mathbf{n}, \mathbf{x} \in \mathcal{U} \subset \mathbf{R}^3$ (the domain $\mathcal{U}$ may coincide with $\mathbf{R}^3$). Let $\{H_i\}$ be k disjoint compact subsets of $\mathcal{U}$ which are called "holes". $\varphi(\mathbf{x})$ is the Gauss map $\varphi : \mathcal{U} \rightarrow S^2$ in the absence of holes. In the presence of holes, consider a spherical neighborhood around some particular H_i. If φ is restricted to this neighborhood, then, with such restriction, φ defines a map $S^2 \longrightarrow S^2$. This map has a degree $d_i \in \mathbf{Z}$ (integers, possibly including zero). If now

$\Omega = \mathcal{U}\backslash(\cup_{i=1}^{k} H_i)$, then one is interested in finding the harmonic map $\varphi : \Omega \to S^2$, that is to find

$$E = \inf_{\varphi \in \varepsilon} \int_{\Omega} \mathrm{d}^3 x (\nabla\varphi)^2 \tag{4.25}$$

under conditions ($i = 1 - k$)

$$\varepsilon = \left\{ \varphi \in (\Omega; S^2) \mid \deg(\varphi; H_i) = \mathrm{d}_i, \quad \int_{\Omega} \mathrm{d}^3 x (\nabla\varphi)^2 < \infty \right\}.$$

The above problem can be formulated as well in two dimensions. In this case, one is talking about the harmonic maps from S^1 to S^1. For d_i which are *integers* and satisfy the condition $\sum_i d_i = 0$ the problem is solved in Ref. [42] with the result:

$$\begin{aligned} E &= \frac{1}{2} \int_{\Omega} \mathrm{d}^2 x (\nabla\varphi)^2 \\ &= -\pi \sum_{i<j} d_i d_j \ln |a_i - a_j| + \text{ boundary term} + \text{const.}, \end{aligned} \tag{4.26}$$

where $\{a_i\}$ play the same role as $\{H_i\}$ in three dimensions. This result provides the desired Coulomb gas analogy used for the description of the defect mediated melting transitions in quasi-two-dimensional liquid crystals [43] via the Kosterlitz–Thouless type of transition which was discussed in Part I. From *the same* physical literature it is known, however [23,37,43] that the defects with integer d_i are topologicaly unstable (at least in R^3) and only those which have the half integer d_i are topologicaly stable.

Remark 4.3. *In two dimensions the degree d_i coincides with the index* I.

Remark 4.4. *In two dimensions, in view of Remark* 4.2, *if we would have stable defects with integer and half-integer* I *we would run into problems since the textures originating from defects with integer* I *produce the orientable (vector) fields on the surface while the defects with half-integer* I *produce the nonorientable (line) fields. According to Strebel's book* [7], *in this case one should consider the field of textures (foliations) coming from defects with integer index* I *as nonorientable line field.* [1]

The quadratic differential (3.1), describes the defect with $I = 1$ and is depicted in Fig. 4b of Part I, while the same differential with $a \to ia$ is also having index $I = 1$ and is depicted in Fig. 2 of Part I. The very existence of the Whitehead moves (Fig. 4, Part I), would be questionable should *both* integer and half-integer fields not be treated as the line fields [7].

As in the case of 2 + 1 gravity (to be discussed in Section 5), because of the nonorientability of the line fields, there is no interaction between the defects but there is an energy, nevertheless. And this energy can be minimized.

[1] In the case of 2 + 1 gravity, (Section 5) such a remark is equivalent to saying that use of quadratic differentials allows to provide a formal unification of the description of electicity and gravity. Such unification is completely different in nature from that proposed by Kaluza and Klein. Please, see also Section 6.2.

Indeed, in two dimensions we have $\mathbf{n} = \{\cos\varphi(\mathbf{r}), \sin\varphi(\mathbf{r})\}$, $\mathbf{r} = \{x, y\}$. This produces, instead of Eq. (4.24), the following result for the distortion energy:

$$\mathcal{F}_{\mathrm{d}} = \frac{\tilde{K}}{2}\int_{\Omega} \mathrm{d}^2 r(\nabla\varphi)^2, \tag{4.27}$$

where the domain Ω is analogous to that defined in Eq. (4.25) (adjusted for 2-dimensional case). The functional $D[\varphi] = (2/K)\mathcal{F}_{\mathrm{d}}$ is known in the literature as Dirichlet integral [44]. This integral has some remarkable properties summarized in the following theorems.

Theorem 4.5. *Let the function $w = f(z)$ provide the conformal mapping of the domain Ω onto Ω^* and let $\varphi(x, y) = \psi(f(z))$, then*

$$D[\varphi] = \iint_{\Omega} (\varphi_x^2 + \varphi_y^2)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\Omega^*} (\psi_u^2 + \psi_v^2)\,\mathrm{d}u\,\mathrm{d}v, \tag{4.28}$$

where $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$ and $\varphi_x = \partial\varphi/\partial x$, etc.

Proof. Indeed, taking into account that

$$\varphi_x = \psi_u u_x + \psi_v v_x, \tag{4.29a}$$

$$\varphi_y = \psi_u u_y + \psi_v v_y, \tag{4.29b}$$

and empoloying the Cauchy–Riemann equations

$$u_x = v_y, \quad v_x = -u_y \tag{4.30}$$

we obtain,

$$\varphi_x^2 + \varphi_y^2 = (\psi_u^2 + \psi_v^2) \cdot (u_x^2 + u_y^2). \tag{4.31}$$

But the Jacobian $J = u_x v_y - u_y v_x = u_x^2 + u_y^2$ in view of Eq. (4.30). Hence, indeed, $D[\varphi]$ is conformal invariant. □

Corollary 4.6. *Taking into account that*

$$(\varphi_x^2 + \varphi_y^2)\,\mathrm{d}x\,\mathrm{d}y = 4|\varphi_z(z)|^2\,\mathrm{d}^2 z \tag{4.32}$$

one has actually more:

$$\iint_{\Omega} \mathrm{d}^2 z|\varphi_z(z)|^2 = \iint_{\Omega^*} \mathrm{d}^2 w. \tag{4.33}$$

That is the Dirichlet integral $D[\varphi]$ defined in Ω is equal to the area of Ω^* which is the image of the area Ω upon the conformal mapping $w = f(z)$.

Theorem 4.7. *Let $w = \varphi(z)$ be the conformal mapping and $\Phi(w)$ be the quasi-conformal mapping, that is, it is performed with help of function Φ such that*

$$\mathrm{d}\Phi = \Phi_w\,\mathrm{d}w + \Phi_{\bar{w}}\,\mathrm{d}\bar{w}, \tag{4.34}$$

then

$$D[\Phi(\varphi(z))] \leq \hat{K} D[\varphi(z)], \tag{4.35a}$$

where the dilatation factor $\hat{K}$ is defined by

$$\hat{K} = \frac{|\Phi_w| + |\Phi_{\bar{w}}|}{|\Phi_w| - |\Phi_{\bar{w}}|}, \quad \hat{K} > 1 \quad \text{if} \quad |\Phi_{\bar{w}}| \neq 0. \tag{4.35b}$$

Proof. Please, consult Ref. [21]. □

In view of Theorem 4.5, our use of isoperimetric inequalities, discussed in Section 2, and quadratic differentials, discussed in Section 3, becomes almost obvious. Since $D[\varphi]$ has the meaning of an area, we can write as well (in view of Eq. (2.29)):

$$D[\varphi] = \iint_{\Omega} \rho^2(\varphi)\, \mathrm{d}x \wedge \mathrm{d}y, \tag{4.36}$$

where now

$$\rho^2(\varphi) = \varphi_x^2 + \varphi_y^2. \tag{4.37}$$

As for the length L_φ, it should be defined accordingly through $|\mathrm{d}w| = \rho|\mathrm{d}z|$. Then, the isoperimetric inequalities discussed in Section 2 can be used immediatly. As for Theorem 4.7, we shall need it later, in Section 6, when we shall discuss some more advanced topics. For the time being, we need to discuss how to obtain ρ from ρ^2 given by Eq. (4.37). To this purpose, following Ref. [8], let us introduce the function $g_z = \frac{1}{2}(g_x - \mathrm{i}g_y)$ and such that $g = u + \mathrm{i}\varphi$. Furthermore, let

$$g_z = \sqrt{\Phi(z)}, \tag{4.38}$$

where the above equation defines Φ. Consider now

$$\iint_{\Omega} |\Phi|\, \mathrm{d}x\, \mathrm{d}y = \iint_{\Omega} |g_z|^2\, \mathrm{d}x\, \mathrm{d}y. \tag{4.39}$$

Consider as well the contour integral along some closed curve $\gamma \in \Omega$ given by

$$\oint_\gamma |\mathrm{Im}(\sqrt{\Phi}\, \mathrm{d}z)| = \frac{1}{2} \oint_\gamma |(\varphi_x - u_y)\, \mathrm{d}x + (u_x + \varphi_y)\, \mathrm{d}y|. \tag{4.40}$$

If the function g_z is analytic (holomorphic), then the Cauchy–Riemann equations produce $u_x = \varphi_y$ and $u_y = -\varphi_x$ which leads us to the conclusion that

$$\oint_\gamma |\mathrm{Im}(\sqrt{\Phi}\, \mathrm{d}z)| = \oint_\gamma |\mathrm{d}\varphi|, \tag{4.41}$$

where $|\mathrm{d}\varphi| = |\varphi_x\, \mathrm{d}x + \varphi_y\, \mathrm{d}y|$. At the same time, the same Cauchy–Riemann equations applied to Eq. (4.39) produce

$$D[\varphi] = \iint_{\Omega} (\varphi_x^2 + \varphi_y^2)\, \mathrm{d}x\, \mathrm{d}y = A_\Omega(\rho). \tag{4.42}$$

Although thus defined construction had provided us with the area $A_{\Omega}(\rho)$, the contour integral

$$h_{\varphi}(\gamma) = \oint_{\gamma} |\,\mathrm{d}\varphi| \tag{4.43}$$

is *not exactly* the φ-length. It is called the *height* (the φ-length is just $\oint_{\gamma} |\sqrt{\Phi}||\,\mathrm{d}z|$). If we are looking for the extremum of (4.42), we cannot apply directly the results of Section 2 because we still have not obtained expression for φ-length. To find out the geometric meaning of the height, it is sufficient to go back to Fig. 1, and to use Remark 3.1 along with Eq. (3.4). Since $(\mathrm{d}w)^2$ in Eq. (3.4) is just the usual Euclidean length element, this means, that in terms of w we have to consider some square, e.g. like that depicted in Fig. 1, with $\operatorname{Im} w = b$ being indeed the height of the square. Eq. (4.41) reflects just this fact. Suppose, as in Fig. 1, we have the annulus and inside of the annulus we have a closed curve which touches both the inner and the outer circle. Then, the image of this curve in w-plane will also be a closed curve which touches the horizontals. This curve, naturally, will have some horizontal and vertical parts. If the height of the rectangle is b, then we obtain,

$$h_{\varphi}(\gamma) = b \leq \int_{b_u} |\,\mathrm{d}\tilde{\varphi}| = \int_{b}^{a} \left| \frac{\partial \tilde{\varphi}(u,\varphi)}{\partial \varphi} \right| \mathrm{d}\varphi, \tag{4.44}$$

where $\tilde{\varphi}(u,\varphi)$ denotes any smooth curve which joins the horizontals and b_u represents the contour. Multiplying both sides of (4.44) by a we obtain,

$$ab \leq \int_0^a \int_0^b \left| \frac{\partial \tilde{\varphi}(u,\varphi)}{\partial \varphi} \right| \mathrm{d}\varphi\,\mathrm{d}u. \tag{4.45}$$

By squaring the above inequality and using the Schwarz inequality (analogous to (2.15)) we obtain,

$$\begin{aligned}(ab)^2 &\leq ab \int_0^a \int_0^b \left(\frac{\partial \tilde{\varphi}}{\partial \varphi}\right)^2 \mathrm{d}\varphi\,\mathrm{d}u \\ &\leq ab \int_0^a \int_0^b \left[\left(\frac{\partial \tilde{\varphi}}{\partial \varphi}\right)^2 + \left(\frac{\partial \tilde{\varphi}}{\partial u}\right)^2\right] \mathrm{d}\varphi\,\mathrm{d}u.\end{aligned} \tag{4.46}$$

Finaly, using Theorem 4.5, we obtain,

$$ab \leq \iint_{\Omega} [\tilde{\varphi}_x^2 + \tilde{\varphi}_y^2]\,\mathrm{d}x\,\mathrm{d}y. \tag{4.47}$$

Of course, in the above derivation we had made a restriction of having just one puncture in Ω. We have effectively surrounded the puncture (the critical point or the singularity) by a circle and considered the domain Ω as an annulus which was converted into the rectangle (as usual). Evidently, the generalization to many singularities should be obvious now. The combination ab can be written therefore as follows:

$$ab = b^2\hat{M} = b^2/M = a^2 M = a^2/\hat{M}, \tag{4.48}$$

where M is given by Eq. (2.26) and $\hat{M} = M^{-1}$ by Eq. (2.35). Using these results we can write our final expression for the distortion energy, Eq. (4.27), for a set of punctures:

$$\mathcal{F}_{\mathrm{d}} \geq \frac{\tilde{K}}{2} \sum_i a_i^2 M_i, \tag{4.49}$$

where a_i is equal to $|I(p_i)|$ as can be seen from reading p. 14 (top) of Ref. [21] in view of the definition of the index, Eq. (4.23). Additional details related to the choice of domains, structure of trajectories of quadratic differentials, etc., could be found in Ref. [45] which does not contain any physical applications, however. Eq. (4.49) coincides with Eq. (5.3) (see also Eq. (2.5)) of Part I where it was given without proof. [2]

Remark 4.8. *The existence of the hexagonally ordered phases discussed in Part* I, *e.g. see Figs.* 8 *and* 10 *of Part* I, *can be easily understood based on the results just obtained. Indeed, let us surround each defect with simple (non-self-intersecting) contour C and consider an area which is enclosed by such contour. Now, the problem can be formulated as follows: for a given perimeter length l of the contour C find a minimal area A which such contour encloses, provided that the obtained figure can cover the surface S without gaps (i.e. tesselates S). For R^2, S^2 and R the results are well known* [46]. *In particular, for R^2 there are only two options: to have squares or to have equilateral triangles. For given perimeter length l the area A of the triangle is smaller than that of the square. Hence, triangles tesselate R^2 under the most optimal conditions and this is the origin of the existence of the hexatic phase. For the alternative (physical) proof of the existence of the hexatic phase, please, consult Ref.* [47].

5. Applications to 2 + 1 gravity

Although in the previous sections we have provided all necessary essentials needed for the description of classical 2 + 1 gravity, here we need some ramifications of the obtained results to facilitate uninterrupted reading.

5.1. Conical singularities and quadratic differentials

The connection between the conical singularities and quadratic differentials was discovered by Troyanov [10]. His derivation is incomplete, however, as was recently noticed by Rivin [11]. Because of this incompleteness, we would like to provide here an alternative derivation of Troyanov's results in order to make connections with those of Rivin.

The main idea of both works lies in the recognition of the fact that any 2-dimensional Riemann surface (and also 3-dimensional manifold [36]) admits consistent triangulation

[2] While this work was under refereeing, Prof. Michael Wolf (Rice U.) had supplied us his latest paper on "Measured foliations and harmonic maps of surfaces" (J. Diff. Geom. 49 (1998) 437–467). It provides much more comprehensive mathematical support of the results discussed above.

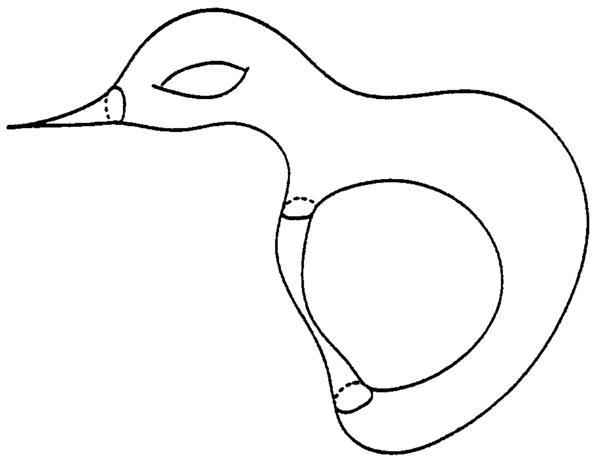

Fig. 6. The thin–thick decomposition of a hyperbolic surface.

with help of *flat Euclidean* triangles. The curvature effects are concentrated then on vertices (cones) of the triangulated surface. If the surface is without boundary (closed), then the curvature singularities are well modelled by the cones on flat Euclidean backgrounds. If the surface has boundaries, then some of the singularities should be modelled by the punctured (truncated) cones [11,48]. Surfaces which have negative Euler characteristics are hyperbolic and, whence, they represent an example of a "hyperbolic paper" (in Thurston's terminology [36]). Since the hyperbolicity is always associated with tractrix-like conical-type surfaces (e.g. see Fig. 6) [17,36] and since such surfaces *cannot* be smoothly embedded into 3-dimensional Euclidean space, e.g. see Theorem 7.1 and Refs. [36,46], the very tip of the cone may be cut off. This creates a boundary (for an illustration, please, consult Ref. [48], especialy p. 98 and 99) since the cone is being truncated now. The Euler characteristic of such triangulated surfaces with truncated cones could be calculated with some effort [11] thus providing the major correction to the results of Troyanov. This correction, naturally, is affecting the results related to $2+1$ gravity as we shall demonstrate shortly below. Consider now the following lemma.

Lemma 5.1. *Suppose we have the quadratic differential which has the infinitesimal length* $\mathrm{d}l^2$ *given by*

$$\mathrm{d}l^2 = |z|^{2\beta}|\mathrm{d}z|^2, \quad \beta \geq -1, \tag{5.1}$$

then, there is a conical metric given by

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}t^2, \quad 0 \leq t \leq \alpha 2\pi, \ \ 0 \leq \alpha \leq 1, \tag{5.2}$$

so that (5.1) *can be mapped into* (5.2) *provided that* $\alpha = 1+\beta$ *and* $\alpha = \theta/2\pi$ *with* θ *being an angle of the cone.*

Proof. We had seen already in Section 3, e.g. see Eqs. (3.9), (3.11) and Fig. 2, that the maping of the type $w = z^{(n+2)/2}$ converts the sector $0 \leq \arg z \leq 2\pi/(n+2)$ in z-plane into the upper w half-plane. If we identify the sides of this sector, we shall obtain a cone with an angle $2\pi/(n+2)$. Clearly, the metric $\mathrm{d}l^2$ in w-plane is Euclidean, i.e. $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$. When going from w to z-plane, we anticipate that the metric will have the form given by Eq. (4.6). To calculate the conformal factor $\hat{\lambda}$, let us consider, in the light of the just presented example, the transformation:

$$x = ar^{\alpha}\cos\alpha\varphi, \qquad y = ar^{\alpha}\sin\alpha\varphi. \tag{5.3}$$

Using these results, we obtain,

$$\begin{aligned} \mathrm{d}x &= a\alpha r^{\alpha-1}\,\mathrm{d}r\cos\alpha\varphi - ar^{\alpha}\alpha(\sin\alpha\varphi)\,\mathrm{d}\varphi, \\ \mathrm{d}y &= a\alpha r^{\alpha}\,\mathrm{d}r\sin\alpha\varphi + ar^{\alpha}\alpha(\cos\alpha\varphi)\,\mathrm{d}\varphi. \end{aligned}$$

Based on these results,we arrive at

$$\mathrm{d}x^2 + \mathrm{d}y^2 = a^2\alpha^2 r^{2(\alpha-1)}(\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2). \tag{5.4}$$

If we demand that $a^2\alpha^2 = 1$, this then allows us to get rid of a. Next, we look at Eq. (5.1) and recall that $|\,\mathrm{d}z|^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2$ and $r = |z|$. Hence, in view of Eq. (5.4), we arrive at the result:

$$\alpha - 1 = \beta. \tag{5.5}$$

Furthermore, if we perform the following rescaling:

$$t = \alpha\varphi, \quad \rho = \alpha^{-1}r^{\alpha}, \tag{5.6}$$

then the metric given by Eq. (5.4) is converted into that given by Eq. (5.2) with $\rho \leftrightarrows r$. Clearly, it makes sense to choose $\alpha = \theta/2\pi$ where θ is the cone angle. Then, Eq. (5.5) produces

$$\theta = 2\pi(\beta + 1) \tag{5.7}$$

in complete agreement with the result of Troyanov [10] where it was obtained in a somewhat different way. □

Corollary 5.2. *By employing the results of Hopf, e.g. see Eqs.* (4.21) – (4.23), *we obtain the index of the quadratic differential:*

$$I(p_i) = 1 - \alpha_i = 1 - \frac{\theta_i}{2\pi} = -\beta_i. \tag{5.8}$$

Corollary 5.3. *The Poincare′–Hopf index theorem, Eq.* (1.9), *Part* I, *can now be written at once as*

$$\chi = \sum_i \left(1 - \frac{\theta_i}{2\pi}\right). \tag{5.9}$$

This result is also in agreement with that obtained in Ref. [10] *where, again, it was obtained differently.*

Since Troyanov is not using Hopf's arguments explicitly, there is no restriction on $I(p_i)$ to be an integer or half-integer (e.g. see Definition 4.1) in his work. We also (only for a moment!) will suppress this restriction in order to discuss some known facts about classical $2+1$ gravity.

5.2. $2+1$ *gravity and quadratic differentials*

In the system of units in which the speed of light $c = 1$, the Einstein's equations are known to be [49]

$$G^{\alpha}_{\beta} = 8\pi G T^{\alpha}_{\beta}; \quad \alpha, \beta = 1 - 4. \tag{5.10}$$

Here G is the gravitational constant, T^{α}_{β} is the energy–momentum tensor and G^{α}_{β} is the Einstein's tensor,

$$G^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \tfrac{1}{2} R \delta^{\alpha}_{\beta}, \tag{5.11}$$

R is the scalar curvature (in the case of two dimensions $R/2 = K$ where K is defined in Eq. (4.4)) and R^{α}_{β} is the Ricci tensor (obtained by contraction from the Riemann curvature tensor). In $2+1$ dimensions in synchronous system of coordinates [50] the first fundamental form is given by

$$\mathrm{d}l^2 = -\mathrm{d}t^2 + \gamma_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j, \quad \mathbf{x} = \{x_1, x_2\} \tag{5.12}$$

(to be compared with Eq. (4.1)).The Einstein tensor G^{α}_{β} has only one nonzero component [9] $G^0_0 = -\frac{1}{2}R$ and, accordingly, T^{α}_{β} also has only one nonzero component T^0_0 given by

$$T^0_0 = -\sum_i m_i \frac{1}{\sqrt{\gamma}} \delta^2 \mathbf{x} - \mathbf{x}_i. \tag{5.13}$$

Multiplying both sides of Eq. (5.10) by $\sqrt{\gamma}$ (where γ is $\det \gamma_{ij}$) and integrating over the surface we obtain,

$$\frac{1}{8\pi G} \int \mathrm{d}^2 x \sqrt{\gamma} K = \sum_i m_i. \tag{5.14}$$

Since $(1/2\pi) \int \mathrm{d}^2 x \sqrt{\gamma} K$ is the Euler characteristic, $\chi = 2 - 2g$, of the surface of genus g, we conclude, that *at least in* $2+1$ *dimensions*, the integrated Einstein equations coincide with the Poincare-Hopf index theorem, Eq. (1.9), Part I. *Conversely, one can arrive at correct Einstein equations starting from the P-H index theorem.* In this case, by the way, there is no need to invoke the equivalence principle as it is traditionally done. Accordingly, there is no need to worry about the justification of the Mach principle [3] since Eq. (5.14) is automatically in accord with it. This becomes especially evident in view of the mass quantization to be discussed below.

By comparing Eqs. (5.9) and (5.14) we obtain at once

$$\sum_i \left(1 - \frac{\theta_i}{2\pi}\right) = \sum_i 4Gm_i \tag{5.15}$$

which produces

$$\alpha_i = \frac{\theta_i}{2\pi} = 1 - 4Gm_i. \tag{5.16}$$

This result is in complete accord with the results of Deser et al. [9] where a somewhat different set of arguments was employed. Taking into account Lemma 5.1, we obtain as well the following result for the metric:

$$\mathrm{d}l^2 = \prod_i |z - z_i|^{2\beta_i} |\,\mathrm{d}z|^2, \tag{5.17}$$

where $2\beta_i = -8Gm_i$ in view of Eqs. (5.5) and (5.16). This result is also in agreement with that obtained in Ref. [9].

Consider now a special case of Eq. (5.14): $g = 0$. Then Eqs. (5.5) and (5.9) produce the following constraint on β_i:

$$\sum_i \beta_i = -2. \tag{5.18}$$

Following Troyanov [10], let us consider the change of variables in Eq. (5.17): $w = z^{-1}$. This produces

$$|\,\mathrm{d}z|^2 = \frac{|\,\mathrm{d}w|^2}{|w|^4}$$

and

$$\prod_i |z - z_i|^{2\beta_i} = \prod_i |w|^{-2\beta_i} |1 - wz_i|^{2\beta_i} = |w|^4 \prod_i |1 - wz_i|^{2\beta_i}. \tag{5.19}$$

Collecting all terms together, we obtain for the metric,

$$\mathrm{d}l^2 = \prod_i |1 - wz_i|^{2\beta_i} |\,\mathrm{d}w|^2. \tag{5.20}$$

This result was obtained with account of the constraint, (5.18). Hence, the metric $\mathrm{d}l^2$ is regular at infinity, $w = 0$, and maintains the same form as given in Eq. (5.17). This result should hold under the restriction that α_i cannot become negative in view of Eq. (5.2). This

[3] According to W. Pauli, "Theory of Relativity" (Dover, NY, 1981), the Mach principle is a postulate "that the inertia of matter is solely determined by the surrounding masses. It must therefore vanish when all other masses are removed..." (p. 179).

happens to be a very serious restriction. Indeed,according to Eq. (5.14) for the case of $g = 0$ and just for one mass m we obtain,

$$1 = 2Gm. \tag{5.21}$$

Using this value of mass in Eq. (5.16) we also obtain,

$$\alpha = 1 - 4G \cdot \frac{1}{2G} = -1. \tag{5.22}$$

This is not permissible, however, since it contradicts Eq. (5.2). Moreover, let $g > 0$ in Eq. (5.14), then we obtain,

$$\sum_i m_i \leq 0, \quad g > 0. \tag{5.23}$$

This result is, apparently, meaningless as well since we expect our masses to be nonnegative.

So far, we have not imposed an additional constraint coming from the Hopf quantization rule, Definition 4.1. If we constrain our indices to the (half) integers, then we obtain the following *mass quantization condition*:

$$4Gm_i = \begin{cases} 1 \\ \frac{1}{2} \end{cases}. \tag{5.24}$$

In the first case we have the degenerate case, $\alpha = 0$, according to Eq. (5.16), and in the second, we obtain, $\alpha = \frac{1}{2}$ which produces prong-type singularity, e.g. see Fig. 2 of Part I which has the index $\frac{1}{2}$.

So far, we have not invoked more recent results obtained by Rivin [11]. If we use his results then, instead of Eq. (5.9), we obtain ($\theta_i > 0$):

$$\chi = \sum_{v_i \notin \partial S} \left(1 - \frac{\theta_i}{2\pi}\right) + \frac{1}{2} \sum_{v_j \in \partial S} \left(1 - \frac{\theta_i}{\pi}\right), \tag{5.25}$$

where the first sum runs, as before, over the conical singularities while the second runs over the truncated (punctured) conical singularities. The above extension of Troyanov's results in spite of its simple look is highly nontrivial and is not straightforwardly obtainable. [4] This result becomes especially useful if we would like to remove the restriction on the total mass of our "Universe", e.g. for $g = 0$ using Eq. (5.15) we obtain,

$$\frac{1}{2G} = \sum_i m_i, \tag{5.26}$$

in accord with Ref. [9]. To remove this restriction, let us consider, for example, how Eq. (5.24) works for a disk D^2 (evidently, S^2 can be obtained by gluing together two disks). If the "boundary sum" is ignored and only the nondegenerate cone angles are considered, then, in agreement with the results of Part I, especially the appendix, we should have two thorn-like singularities using the quantization condition (5.23) and the fact that for

[4] Nevertheless, it could be found already in Thurston's lecture notes (Ref. [20], Part I), e.g. see p. 13.20.

the disk $\chi(D^2) = 1$. We can place yet another thorn on D^2 if we use the "boundary sum" term. Without violating the constraint $\theta_i > 0$ and requiring θ_i/π to be a positive integer [13], we obtain the first nontrivial result : $\theta_i/\pi = 2$, which produces the desired $-\frac{1}{2}$ factor characteristic of Y-type defects (Fig. 2, Part I). The rest of the arguments used in Part I now can go through without change so that the restriction for the total mass can now be removed. Evidently, the extension of these results to higher genus surfaces becomes also possible without any problems.

Remark 5.4. *At this point it is appropriate to remind the reader that even for the Coulombic charges on S^2 the topology requires only two charges in order for the $P - H$ index theorem to be satisfied (e.g. see Section* 1, *Part* I). *This is completely analogous to the total mass restriction, Eq.* (5.25), *for the case of gravity. The electroneutrality requirement emerges naturally if we want to put the additional charges on S^2. Moreover, although we actually can put only the "sources" ("+") and the "sinks"("−") on S^2, the presence of these additional singularities (both having index* $+1$) *automatically creates the induced saddles (with index* -1). *Something similar occurs in the case of gravity because of an extra boundary sum term (Rivin (or, rather, Rivin–Thurston) sum term).*

Remark 5.5. *The need of such boundary term(s) has deep physical meaning (see "Note added in proof" at the end of this paper) and can be readily appreciated if one would like to use Eq.* (5.14) *for surfaces of genus g higher (or equal) than one. Since for $g = 1$ the l.h.s. of* (5.14) *is 0, we obtain: $\sum_i m_i = 0$ ("electroneutrality") which is of limited use (photons, neutrinos, etc.) for gravity. For $g > 1$ we obtain $\sum_i m_i < 0$ and this is physically problematic as was noticed already by Einstein.*

Remark 5.6. *In the original work by Troyanov* [10] *there is no restriction on α_i to be in the range between* 0 *and* 1. *It can be any positive integer or half-integer (Definition* 4.1). *Under such conditions the relation given by Eq.* (5.9) *becomes correct for surfaces of any genus. Unfortunately, it cannot be used for gravity for reasons explained in the previous remark.*

Remark 5.7. *It is useful to recall why altogether one should be concerned with surfaces of higher genus in the case of* $2 + 1$ *gravity. Following Petrov* [51], *Einstein spaces are characterized by the condition:*

$$R_{ij} = \tilde{\lambda} g_{ij}. \tag{5.27}$$

Since the scalar curvature $R = R^i_i = g^{ik} R_{ik}$, the constant $\tilde{\lambda}$ in Eq. (5.26) *can be eliminated with the result:*

$$R_{ij} = \frac{R}{d} g_{ij}, \tag{5.28}$$

where d is the dimensionality of space. Using this result, the Einstein tensor (5.11) *can be rewritten as*

$$G^\alpha_\beta = \left(\frac{1}{d} - \frac{1}{2}\right) \delta^\alpha_\beta R \tag{5.29}$$

so that the covariant derivative of both sides of Eq. (5.10) *produces*

$$G^{\alpha}_{\beta,\gamma} = 0, \tag{5.30}$$

since $T^{\alpha}_{\beta,\gamma} = 0$ *by construction. Eq.* (5.29) *can be equivalently rewritten* (*with account of Eq.* (5.28)) *as*

$$\left(\frac{1}{d} - \frac{1}{2}\right) R_{,\gamma} = 0. \tag{5.31}$$

For $d \neq 2$ *we have to study spaces of constant curvature called Einsein spaces.For* $d = 2$ (*that is for the fixed time slice*) *we have no choice but to conclude that any 2-dimensional surface is Einstein space. But any 2-dimensional surface is the Riemann surface* [46].

6. Some more advanced topics: a brief discussion

6.1. Inclusion of the cosmological term

Einstein's equation (5.10) is written without the cosmological constant Λ term. This deficiency can be easily corrected. By multiplying both sides of Eq. (5.10) by $\sqrt{\gamma}$ and taking into account that [9]

$$-\sqrt{\gamma}G^0_0 = \tfrac{1}{2}\sqrt{\gamma}R \tag{6.1}$$

and $R/2 = K$ (Eq. (4.4)) where, for the metric given by $\mathrm{d}l^2 = \rho|\,\mathrm{d}z|^2$, the Gauss curvature K is known to be [52]

$$K = -\frac{2}{\rho}\frac{\partial^2 \ln \rho}{\partial z \partial \bar{z}}. \tag{6.2}$$

Using these results along with Eqs. (5.10) and (6.2) we obtain,

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + \frac{\Lambda}{2}\exp\varphi = -8\pi G \sum_i m_i \delta^2(z - z_i), \tag{6.3}$$

where $\rho = \exp\varphi$. For $\Lambda = -1$ and $8Gm_i = 1$ the above equation coincides with the Liouville equation discussed by Takhtadjian [14] in connection with the nonperturbative approach to string theory. Accordingly, quantization of $2+1$ gravity and $1+1$ string theory are related to each other as was noticed by Witten [53]. Consider now the Liouville equation in the punctured complex plane that is in $\mathbf{C}\backslash\{z_1, \ldots, z_n\}$, and let Λ be some *positive* constant. This case (actually corresponding to the punctured sphere S^2) was treated in Ref. [54]. The authors of Ref. [54] were able to prove the following theorem.

Theorem 6.1. *The Liouville equation* $\nabla^2\varphi = -\exp(2\varphi)$ *in the punctured complex plane* $+\{\infty\}$ *has solution. Near each puncture the solution is represented by* $\varphi(z) = \beta_i \ln|z - z_i| +$ *harmonic function. The constants* $\beta_i = (\theta_i/2\pi) - 1$ *and satisfy the restriction:*

$$\sum_i \beta_i \geq -2 \tag{6.4}$$

and

$$\left(\sum_{i\neq j}\beta_i\right) - \beta_j < 0 \quad \text{for all} \quad j = 1, \dots, n. \tag{6.5}$$

Remark 6.2. *Although the methods used in Ref.* [54] *are different from those used by Troyanov* [10] *and Rivin* [11], *the results are, actually, the same. It appears, therefore, that one can use their results to bypass the solution of the Liouville equation* (6.3) (*at least for the case of* S^2).

In the light of this remark, one may think, that a similar proof can be obtained for $\Lambda < 0$ as well. This is, however, not the case due to the results of McOwen [55]. His results are analogous to that discussed by Takhtadjian [14]. Moreover,the same results were obtained much earlier by Nevanlina (e.g. see p. 249 and 250 of Ref. [56] and take into account the very minor typos, e.g. sign errors, etc.). The difference between treatments of $\Lambda > 0$ and $\Lambda < 0$ cases deserves further study.

6.2. *Some sum rules*

Although the Poincare–Hopf index theorem plays the major role in study of topology of 2-dimensional surfaces, sometimes, additional ramifications are required. Let us begin with the following lemma.

Lemma 6.3. *If the quadratic differential has poles with the total order p and zeros (including their multiplicity) of total order q, then for the Riemann surface of genus g with n boundary components we must have*

$$p - q = 4 - 4g - 2n. \tag{6.6}$$

Proof. Consider first the case of $n = 0$. Then, taking into account Eqs. (4.21)–(4.23)and using the $P - H$ theorem, Eq. (1.9), Part I, we obtain,

$$\sum_{\text{poles}} n_i - \sum_{\text{zeros}} n_j = 2(2 - 2g).$$

Let us now have n boundary components. These can be obtained in the following way. Consider a sphere S^2 with g handles and add one additional handle so that the genus of the surface becomes $g + 1$. Let us now squeeze the additional handle so that it breaks into two pieces leaving two punctures on S^2. Hence, we obtain the correspondence: $g \leftrightarrows 2n$. According to Jenkins [57] (see also Rivin [11]) the *boundary* zeros are necessarily of *even order*. Therefore, instead of having $2(2 - 2g) - 4n$ we obtain the result (6.6). □

Because the quadratic differentials are transforming like tensors of rank 2, e.g. see Eq. (3.2), one may ask a question: under what conditions can they be formed out of the product of the abelian differentials? Following Ref. [13], for the Riemann surface R, let k_i

be the order of zero at some $p_i \in R$ and let, for example, $k_i = -1$ if at p_i we have a simple pole (a prong-like singularity), etc., then each quadratic differential ϕ is associated with the set of data $(k_1, \ldots, k_n; \varepsilon)$ where $\varepsilon = +1$ if ϕ is the square of the abelian differential and $\varepsilon = -1$ if it is not. One can prove the following theorem.

Theorem 6.4. *Let $k = (k_1, \ldots, k_n; \varepsilon)$ where $k_i \in \{-1, 1, 2, \ldots\}$ and $\varepsilon = \pm 1$. Then, for a closed surface R there is ϕ realizing k if and only if:*
(a) $\sigma(k) = 0 \bmod 4$, $\sigma(k) \geq -4$ *and* $\sigma(k) = \sum_{i=1}^{n} k_i$;
(b) $\varepsilon = -1$ *if any k_i is odd; and*
(c) $(k_1, \ldots, k_n; \varepsilon) \neq (4; -1), (1, 3; -1), (-1, 1; -1)$ *or* $(\ ; -1)$.

Proof. Please, consult Ref. [13]. □

In case if we are interested under what conditions the pseudo-Anosov regime of surface homeomorphisms is possible, the theorem given below could be also of major importance. To formulate this theorem we have to provide the following definition first.

Definition 6.5. The 1-prong is thorn-like (a simple pole) singularity, the 3-prong is Y-type (a simple zero) singularity. The p-prong is the singularity with p arms [24]. The data set could be realized in terms of the prong numbers p_i: $(p_1, \ldots, p_n; \varepsilon)$.

Theorem 6.6. *There is a pseudo-Anosov homeomorphism on R of genus g and n punctures realizing the data $(p_1, \ldots, p_j; \varepsilon)$ if and only if:*
(a) $\sum_{i=1}^{j} (p_i - 2) = 4(g - 1)$,
(b) $\varepsilon = -1$ *if any p_i is odd,*
(c) $(p_1, \ldots, p_j; \varepsilon) \neq (6; -1), (3, 5; -1), (1, 3; -1)$ *or* $(; -1)$.
(d) *the number of indices* I *for which $p_i = 1$ is less than or equal to n (that is the thorns may be "sitting" at the punctures).*

Proof. Please, consult Ref. [13]. □

Remark 6.7. *Theorems* 6.4 *and* 6.6 *provide some additional selection rules which should be taken into account when the mass spectrum of* $2 + 1$ *gravity is of interest. They do not follow trivially from the $P - H$ theorem.*

6.3. Connections with the theory of Teichmüller spaces

In this subsection we would like to discuss briefly under what *physical* conditions it is possible to anticipate the transitions from the periodic to the pseudo-Anosov regime.These results are complementary to that presented in Section 5 of Part I.

We have encountered a glimpse of the Teichmüller theory in Section 4.2. From it,we realize that we are in the domain of the Teichmüller theory as soon as instead of the conformal mapping (for which $\Phi_{\bar{w}} = 0$) we are dealing with the quasi-conformal (for which $\Phi_{\bar{w}} \neq 0$). The length element given by Eq. (4.1) can be brought into form [58]

$$\mathrm{d}l^2 = \Lambda(z, \bar{z}) |\mathrm{d}z + \mu(z)\, \mathrm{d}\bar{z}|^2, \tag{6.7}$$

where

$$\Lambda = \tfrac{1}{4}(E + G + 2\sqrt{EG - F^2})$$

and

$$\mu = \frac{E - G + 2\mathrm{i}F}{E + G + 2\sqrt{EG - F^2}}$$

in notations of Section 4.1. At the same time, the length element given by Eq. (4.1) can be brought into the conformal form given by Eq. (4.6). Using Eq. (4.6), let $w = x_1 + \mathrm{i}x_2 \equiv u + \mathrm{i}v$. Then, we have as well,

$$\mathrm{d}l^2 = \rho|\,\mathrm{d}w|^2 = \rho(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{6.8}$$

If now $w = w(z, \bar{z})$, we obtain,

$$|\,\mathrm{d}w|^2 = |w_z|^2 \left| \mathrm{d}z + \frac{w_{\bar{z}}}{w_z}\,\mathrm{d}\bar{z} \right|^2, \tag{6.9}$$

where, as before, $w_z = (\partial/\partial z)w$, etc. Comparison between Eqs. (6.7) and (6.9) produces

$$\Lambda(z, \bar{z}) = \rho|w_z|^2. \tag{6.10}$$

This result coincides with Eq. (3.2) as required. Comparison between Eqs. (6.7) and (6.9) produces as well

$$\mu(z) = \frac{w_{\bar{z}}}{w_z}. \tag{6.11}$$

This is known as *Beltrami* equation. If $w_{\bar{z}} = 0$, then $\mu = 0$ and the mapping is conformal. Looking at Eq. (4.35) we can write as well [29]

$$|\mu(z)| \le \frac{\hat{K} - 1}{\hat{K} + 1} < 1. \tag{6.12}$$

Either $|\mu|$ or $\hat{K}$ determine the *amount of stretching* the surface undergoes (this strething can include twisting as well).

Let us now introduce notations:

$$\mu(z) \leftrightarrows \mu_w(z)$$

if μ is defined through Eq. (6.11),

$$K_w = \frac{1 + |\mu_w(z)|}{1 - |\mu_w(z)|} = \frac{1 + k}{1 - k}, \quad 0 < k < 1. \tag{6.13}$$

Using these notations one can prove the following theorem

Theorem 6.8. *Let φ be some quadratuc differential on R, then*

$$\mu_w(z) = k\frac{\bar{\varphi}(z)}{|\varphi(z)|}, \tag{6.14}$$

where $\bar{\varphi}$ means the complex conjugate.

Proof. Please, consult Ref. [59]. □

Definition 6.9. A diffeomorphism (possibly with isolated singularities) $f : R \longrightarrow R'$ is admissible if

$$K_f \equiv K[f] < \infty. \tag{6.15}$$

Theorem 6.10. *In order for diffeomorphism to be admissible, we have to require*

$$K[f \circ f] = [K[f]]^2. \tag{6.16}$$

Proof. Please consult Ref. [60]. □

Corollary 6.11. *Using Theorem* 6.8 *it is straightforward to show that if* $|\mu_f(z)| = k$ *(see Eq.* (6.14)), *then* $|\mu_{f\circ f}(z)| = 2k/(1+k^2)$. *By combining Eqs.* (6.13)–(6.16) *we observe that if* $K[f]$ *was initially* > 1, *then*

$$K[f \circ f] = [K[f]]^2 > K[f] > 1.$$

Hence, successive dilatations lead to the successive stretching. This fact produces the most profound implications reflected in the following theorem and corollaries that follow.

Theorem 6.12. *Let* φ *be associated with the admissible map* f *of* R. *If* φ *has only a finite number of critical points, then* φ *has no saddle connections (critical trajectories running between these critical points). In particular, every trajectory of* φ *is dense in* R

Proof. Assume that after some iteration

$$f^m = \underbrace{f \circ \cdots \circ f}_{m}$$

we had been able to fix all saddle connections. Then, on one hand, the length between them should remain the same as it was initially and, on another hand, in view of Eq. (6.16) the length has expanded by the factor of $K^{m/2} > 1$. This contradiction proves the theorem. Consequently, every trajectory of φ is dense in R (otherwise, there would be saddle connections). □

Corollary 6.13. (a) *Theorem* 6.12 *indicates that the existence of the meandritic labyrinths, discussed in Part I, is possible only if the surface undergoes some stretching. Hence, the stretching is needed to destroy the reducible and periodic phases and to create the pseudo-Anosov phase.* (b) *If the Riemann surface* R *has some boundaries which are pointwise fixed, then there is no admissible mapping in any homotopy class of self-mappings. That is the nonslip boundary conditions must be violated* [61, 62] *in order for the pseudo-Anosov phase to become possible. (This is in complete agreement with Thurston's "earthquakes" mentioned in Part* I). (c) *The deformations of* R *may take place in real time. The evolution of such surface is described by the product* $R \times [0, t]$ *which is just some* 3*-manifold (see next subsection). This type of manifold was suggested by Witten in connection with* $2 + 1$ *gravity*

[53] (*without matter fields*). *Inclusion of the matter fields produces the pseudo-Anosov foliations evolving in real time.*

6.4. $2+1$ *gravity and hyperbolic* 3*-manifolds*

In this subsection and in Appendix A, without any pretence on completeness, we would like to mention several facts related to 3-manifolds. More detailed analysis would require a separate publication(s) and is left for the future.

There are several ways to construct 3-manifolds. For the comprehensive treatment of this subject, e.g. see Refs. [63, 64]. A quick introduction into this field is beautifully presented in Refs. [36,65]. In particular, following Ref. [65], we would like to describe one of the methods of constructing 3-manifolds which is naturally connected with the results of the preceding subsection.

Let S be a compact surface (2 manifold), perhaps, with boundary. Let $I = [0, 1]$ be the closed interval of real numbers from 0 to 1. Then, the Cartesian product $S \times I$ is 3-manifold with boundary. Consider now the bottom, i.e. $S \times \{0\}$, and the top, i.e. $S \times \{1\}$, surfaces and let us glue them together. The operation of gluing is trivial if the top surface is the same as the bottom surface (that is if the bottom surface was parallel translated to the top). But, the situation becomes more intresting if one considers the surface homeomorphisms $h : S \longrightarrow S'$. These homeomorphisms are superimposed with the gluing operation in which each point $(x, 0) \in S \times \{0\}$ of the bottom surface is identified with the corresponding point $(f(x), 1) \in S \times \{1\}$ of the top surface. The result of such an identification is a new 3-manifold which is actually a fiber bundle over the circle S^1 (compare this discussion with that of Section 3, Part I) each of the fibers being the original surface at given instant of "time". The fact that a thus constructed manifold is a fiber bundle is in accord with the discussion presented in Thurston's book (Ref. [36], p. 159) where some additional details could be found.

Although we just had explained the construction of a typical 3-manifold, it is still unclear from this construction why this manifold has to be hyperbolic and why $2+1$ gravity should be related to the hyperbolic 3-manifolds(as well as to knots and links).

The connection between 3-manifolds and knots and links was established by Lickorish [66] and Wallace [67] who proved the following theorem.

Theorem 6.14. *Every closed, connected, orientable* 3*-manifold is obtained from* S^3 *by removing mutually exclusive (but, perhaps, knotted and linked) collection of solid (framed) tori* $\{T_i\}$, $i = 1-n$, *and then sewing them back into* S^3 *in a different way.*

From Theorem 6.14. it is not immediately clear how 3-manifolds constructed by Dehn surgery methods used in the theorem by Lickorish and Wallace are related to the 3-manifolds constructed with help of fiber bundle methods. Leaving the answer to this question aside, e.g. see Ref. [20] of Part I for more details, we arrive at the following corollary.

Corollary 6.15. *Every closed, connected, orientable* 3*-manifold is a complement of some knot or link.*

Having this corollary, the question arises: under what conditions the dynamics of 2 + 1 gravity and /or dynamics of liquid crystals can be associated with knots/links or, alternatively, with 3-manifolds (obtained with help of Dehn surgery or fiber bundle methods)?

In 1970 Geroch [68] proved the following theorem.

Theorem 6.16. *If the open set N is globaly hyperbolic, then if it is considered as a manifold, it is homeomorphic to* $\sum^{(3)} \times R$ *where* $\sum^{(3)}$ *is some 3-dimensional manifold and* $\forall a \in R$ *the product* $\sum^{(3)} \times \{a\}$ *is the Cauchy surface.*

Remark 6.17. *For the precise definition of the Cauchy surface, please, consult Ref.* [69].

Remark 6.18. *Theorem* 6.16. *admits natural extension to the case of* 2 + 1 *gravity. In this case we have to replace* $\sum^{(3)}$ *with* $\sum^{(2)}$ *which is the Riemann surface of some finite genus. This is in complete accord with the results of Witten* [53].

The products $\sum^{(2)} \times \mathbf{R}$ had been recently discussed in Ref. [70] (and references therein). In Ref. [70] no attempts were made towards physical applications. Below, and, in part, in Appendix A we shall provide some condensed summary of the results mainly associated with Ref. [70], which could be useful for description of dynamics of 2 + 1 gravity.

Let us return to Theorem 6.16 in order to explain the meaning of words "globally hyperbolic". From Part I we know already about the hyperbolic Poincare disk D^2 and the hyperbolic half plane H^2-model (Section 4). These results can be easily extended. For example, the hyperbolic H^3-model can be defined by analogy with Eq. (4.1) (Part I):

$$H^3 = \{(x, y, t) \in \mathbf{R}^3,\ z = x + \mathrm{i}y = \mathbf{C},\ t > 0\} \tag{6.17}$$

and, instead of the open disk D^2 model defined by Eq. (4.2), now we have to consider an open ball B^3 with sphere S^2_∞ at infinity replacing the circle S^1_∞ at infinity for D^2. Instead of the Fuchsian groups $\mathrm{SL}(2, \mathbf{R})/\{\pm I\}$, we need to use now the quasi-Fuchsian (or Kleinian) groups Γ generated by $\mathrm{SL}(2, \mathbf{C})/\{\pm I\}$.

Definition 6.19. A hyperbolic 3-manifold M is a quotient $M = H^3/\Gamma$ of the hyperbolic H^3 space by the Kleinian group Γ.

Theorem 6.20. *Let* $h : S \longrightarrow S$ *be the pseudo-Anosov homeomorphism. Then, the 3-manifold obtained by fibering over the circle* S^1 *is hyperbolic and has finite (hyperbolic) volume.*

Proof. Please, consult Ref. [70]. □

Corollary 6.21. (a) *Since the volume of such designed 3-manifold is finite, it may be associated with knots (links). That is it can be shown that such manifold is a complement (in* S^3*) of some fibered knot or link (Corollary* 6.15*).* (b) *If we take the hyperbolic 3-manifold that fibers over the circle, with fiber being a 2d surface, and unwrap the covering space by unwrapping the circle direction, then the volume of thus obtained manifold is infinite.*

Proof. Please, consult (a) Ref. [36], (6) Ref. [36], p. 258 □

Remark 6.22. *The result just presented is in accord with Ref.* [69] (*e.g. see paragraph* 6.4). *That is, the physical space–time is noncompact and is a universal covering space of the space–time which is compact.To prove the existence of such compactified space is highly nontrivial. Accordingly, the existence of knots and links in* $2 + 1$ *gravity is not self-obvious.*

Remark 6.23. *Theorem* 6.20 *is also in accord with Theorem* 6.12. *discussed earlier. That is, the surface motion which is associated with stretching, which takes place in real time, is responsible for creating the hyperbolic* 3-*manifolds.*

Remark 6.24. *The previous remark allows us to provide, yet another, interpretation of the process of compactification through the notion of the universal Teichmüller curve which was introduced by Bers* [71] *and known in physical literature in connection with problems related to string theories* [72]. *This and related topics are summarized in Appendix A.*

Remark 6.25. *Although the theory of knots and links was advocated for gravity for some time, e.g. Ref.* [73], *the arguments presented above indicate the special role of fibered knots and links in* $2 + 1$ *gravity. Ref.* [74] *contains prescriptions for construction of such knots and links.*

7. Discussion

7.1. Quantization of $2 + 1$ *gravity*

The results presented in Parts I and II provide mainly classical description of $2+1$ gravity. The quantum description may require computations which are based on the master equation, Eq. (4.13), of Part I. That is one has to consider some sort of the random walk on the mapping class group of some 2-dimensional surface of genus $g > 1$, possibly with punctures. To make these computations meaningful, it is essential to find a physically unambiguous way by which the transition amplitudes W_{ij} in Eq. (4.13) of Part I can be calculated. To this purpose, we anticipate, that the physical results obtained by 't Hooft, e.g. see Refs. [75–78], and their development by Franzosi and Guadagnini [79] could be very helpful. In addition,the results of Appendix A suggest that the quantization could be achieved as well by methods of noncommutative geometry applied directly to the Teichmüller space (more correctly, to the Bers slice). According to Ref. [80], "The quasi-conformal charts provide enough analysis to "quantize the manifold" in the sense of constructing a Hilbert space and relevant operator replacing curvature... The involved Hilbert space theoretical data are of the same nature as those appearing in the transfer matrix theory of statistical mechanics and suggest a purely combinatorial approach...in the extended context of spaces with singularities." In the physical language, the transfer matrix approach discussed in Ref. [80] can be easily recognized as Feynman–Wiener–Kac-type of path integral calculations.

7.2. *Connections with the theory of dislocations and disclinations*

Connections between 2 + 1 gravity and the theory of defects and textures in solids was recently discussed by Katanaev and Volovich, Ref. [15] (Part I). These authors had noticed that the theory of wedge and edge dislocations as well as the linear disclinations in solids is *isomorphic* to the theory of 2 + 1 gravity of Deser et al. Using arguments which are completely different from those presented in Sections 4–6 of Part II, Katanaev and Volovich nevertheless had arrived at correct mass (Hopf) quantization condition, Definition 4.1. They missed, however, the sum rules of our Section 6 which could be obtained only with help of the theory of quadratic differentials. [5]

7.3. *Connections with the theory of motions in classical and quantum billiards*

The problem of quantization of 2 + 1 gravity (as well as motion of point-like defects in the presence of wedge and edge dislocations, etc.) could be considered also from the point of view of classical and quantum motion of particles in billiards. The connection between the quadratic differentials and billiards is known to matematicians for some time, e.g. see Refs. [81–83]. The related literature, e.g. Refs. [61,84] as well as Ref. [85], may be helpful as well. Since in the mathematical physics literature currently there is a strong interest in the detailed study of various mesoscopic systems [86], we anticipate that some "cross fertilization" between different domains of the same discipline (physics) could produce some unexpected results.

7.4. *Crumpling, fracture, brittleness*

In his book, Ref. [36], Thurston discusses the construction of what he calls a "hyperbolic paper". This construction is based on the famous theorem by Hilbert.

Theorem 7.1. *There is no complete smooth surface embedded in Euclidean* 3-*space with the local intrinsic geometry of pseudosphere* (*e.g. see the "beak of the bird " in Fig.* 6).

That is when the surface crumples, it necessarily develops the conical-like "beaks". The mathematics of this crumpling process is discussed in Thurston's famous lecture notes (e.g. see Ref. [20], Part I) and, surely, involves the train tracks. At the same time, in physics literature the process of crumpling of surfaces was recently discussed in Refs. [87,88]. The process of crumpling is closely related to brittleness and fracture. Depending upon the rigidity of surface, it may or may not "want" to crumple under the applied stresses. In the last case, we may observe some cracks (more exactly, the crack patterns, e.g. see Ref. [89], which eventually cause the disintegration of surface. The dynamical equations for such crack patterns are similar to the master equation, (4.13), Part I, for 2 + 1 gravity. Evidently,

[5] Very recently, Katanaev and Volovich had published another paper, "Scattering on dislocations and cosmic strings in the geometric theory of defects" (Ann. Phys. 271 (1999) 203–232). It contains many additional interesting applications.

one can get some additional insights into both fields if one is aware about the existence of the other. Since microscopically the cracks are caused by dislocations/disclinations, e.g. see Section 7.2, surely, one can think about the dynamics of $2 + 1$ gravity in terms of the dynamics and topology of fracture [89–91].

7.5. *Meanders, the Temperley–Lieb algebra and the invariants of 3-manifolds*

In Part I, Section 5, Fig. 25, we have mentioned that any meander (or disconnected system of meanders) can be built by superposition of two arc configurations of the same order: one is being considered as the top while another as the bottom. In Ref. [36], Part I, it is shown that *both* the top and the bottom configurations can be obtained with help of the product of the Temperley–Lieb (TL) algebra generators (since these generators admit braid-like graphical representation) $\{e_i\}$ which obey the TL algebra $\mathrm{TL}_n(\delta)$ given by:

$$
\begin{aligned}
&e_i e_j = e_j e_i \quad \text{if } |i-j| > 1, \\
&e_i^2 = \delta e_i, \quad i = 1, \dots, m-1, \\
&e_i e_{i\pm 1} e_i = e_i, \quad i = 1, \dots, m-1,
\end{aligned}
$$

where δ is some constant. For the TL algebra composed of n elements, $1, e_1, \dots, e_{n-1}$, the number of possible *independent* products of e_i generators is given by the Catalan number, Eq. (3.9), Part I. Following Lickorish [92,93], one can think of space of these independent products as a vector space V_n of dimension C_n. Then, one can construct a bilinear form

$$\langle\, ,\, \rangle : V_n \times V_n$$

which in ordinary language means just a kind of a pairing between the top and the bottom arc configurations as discussed above. As it was shown by Lickorish, the above bilinear form plays the central role in constructing the algebraic invariants of 3-manifolds. Hence, use of the meanders allows to provide the alternative (to the Chern–Simons [94]) formulation of the invariants of 3-manifolds. Additional connections between the TL algebra and the meanders could be found based on the notion of parenthesis [95]. The parenthesis are also associated with TL algebra. By definition, a parenthesis diagram is a word in the alphabet with three letters • (and), e.g. see p. 543 of Ref. [95]. Exactly the same three letter alphabet is used for the description of meanders, e.g. see p. 121 of Ref. [34] (Part I). We believe, that these connections deserve further study.

Note added in proof. After this work was completed, we run across the only one article by Einstein written for the Scientific American (Sci. Am. 182 (4) (1950) 13–17) in which he wrote, e.g. see p. 16 (bottom), "The fact that the masses appear as singularities indicates that these masses themselves cannot be explained by symmetrical g_{ik} fields, or 'gravitational fields'. Not even the fact that only *positive* gravitating masses exist can be deduced from the theory. Evidently a complete relativistic field theory must be based on a field of more complex nature...." In the above, the word "positive" was emphasized by Einstein himself. We were mystified by the fact that the theory of quadratic differentials had begun its development just about the time when Einstein wrote these words.

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Appendix A

In this appendix we would like to provide an outline of the results recently obtained in mathematical literature which are related to the problem of compactification discussed in Section 6.4. The purpose of such presentation also lies in demonstration of nontriviality of the problem of knots/links relevance to 2 + 1 gravity.

Let us begin with the following observations. The group of Mobius transformations Γ with *real* coefficients, PSL (2,R), is defined through the transformation law

$$\gamma(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{A.1}$$

The discontinuous subgroups (e.g. see Definition A.4.) of Γ are called Fuchsian groups. These subgroups are classified as *elliptic* (respectively, *parabolic* and *hyperbolic*) if $|a+d| < 2$ (respectively, $|a+d| = 2$ or $|a+d| > 2$). Compare with appendix to Part I.

A parabolic transformation is conjugate in Γ to the translation: $z \to z + 1$, a hyperbolic transformation is conjugate to dilatation: $z \to \lambda z$ for some $\lambda > 1$. Let $G \in \Gamma$ and let $\pi : H^2 \to H^2/G$ be some Riemann surface, then one can prove the following theorem.

Theorem A.1. *If H^2/G is compact, then each $\gamma \in G\backslash\{id\}$ is hyperbolic*

Proof. Please, consult Ref. [29]. □

These results can be generalized to higher dimensions. For example, instead of H^2 we can use H^3 defined by Eq. (6.17), instead of PSL$(2, R)$ we have to use PSL$(2, C)$. Theorem A.1 is extendable to H^3 and is most useful if instead of H^3 model we would use the Poincare sphere model. That is instead of an open disc D^2 model with S^1_∞ being a "circle at infinity", we would use an open ball B^3 model with S^2_∞ being a "sphere at infinity". The hyperbolic transformations in D^2 have fixed points on S^1_∞. The hyperbolic transformations in B^3 have fixed points on S^2_∞. The hyperbolic polygon in H (or D^2) is being replaced now with the hyperbolic polyhedron in H^3 or B^3.

Definition A.2. A Fuchsian group Γ is finitely generated if the corresponding hyperbolic polygon in H^2 (or D^2) has finite hyperbolic area.

Definition A.3. A quasi-Fuchsian (Kleinian) group is geometrically finite if the corresponding hyperbolic polyhedron in H^3 (or B^3) has finitely many faces.

Remark A.4. *All known knots/links (except torus and satellite) have as their complements the hyperbolic manifolds associated with the geometrically finite Kleinian groups* [36].

Let $G \in \Gamma$ be some element of geometrically finite Kleinian group. Consider the hyperbolic distance $d_G(x) = d(x, G(x))$. Thurston [36] proved the following theorem.

Theorem A.5. *G is hyperbolic if and only if the infimum of d_G is positive. This infimum is attained along the line, which is unique axis for G. The endpoints of the axis are the fixed points of G on S^2_∞.*

Corollary A.6. *If $x \in H^3$ (or B^3), the limit set $L_G \subset S^2_\infty$ is the set of accumulation points of the orbit G_x of x. L_G is independent of the choice of x.*

Definition A.7. The domain of discontinuity D_Γ for a discrete group Γ is given by $D_\Gamma = S^2_\infty \backslash L_\Gamma$. A discrete subgroup of PSL(2, C) acting on D_Γ whose domain of discontinuity is nonempty is called Kleinian group.

Definition A.8. A group Γ acting on a locally compact space X is called properly discontinuous if for every compact set $K \in X$ there are only finitely many $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$.

Definition A.9. The Kleinian manifold $\bar{M}$ is defined by

$$\bar{M} = (H^3 \cup D_\Gamma)/\Gamma.$$

Let S be compact oriented surface of negative Euler characteristic. Let ∂S denote the boundary of S and int $S = S \backslash \partial S$ be the interior of S. Analogously, for 3-manifolds we have the following definition.

Definition A.10. The boundary of $\bar{M}$ is given by $\partial \bar{M} = D_\Gamma / \Gamma$.

Corollary A.11. *Some Kleinian manifolds are bounded by a pair of Riemann surfaces X (bottom) and Y (top). There is a homeomorphism between $\bar{M}$ and* int$S \times [0, 1]$ *which is compatible with marking of X by S and Y by $\bar{S}$ (where $\bar{S}$ is the same surface as S but with reversed orientation).*

Proof. Please, consult Refs. [70,71,96]. □

Remark A.12. *The marking of Riemann surface X should be understood in the following sense. The Teichmüller space $Teich(S)$ classifies the conformal structures on* int$S$ *in which* *each boundary component corresponds to a puncture. A point in* $Teich(S)$ *is specified by X which is conformaly equivalent to a punctured disk* D^2 *(surely, in general, with more than one puncture). A homeomorphism* $f : \text{int}S \to X$ *which sends the orientation on S to the (canonical) orientation on X is called marking of X.*

Corollary A.11. leads us to the most important generalization due to Bers [71]. It can be formulated in the form of the following theorem.

Theorem A.13. *Given two (in general, different) Riemann surfaces X and Y of genus g and an orientation reversing map J between them, there is a uniquely determined (up to normalization) quasi-Fuchsian group G for which the bottom surface is (conformaly equivalent to) X, the top, to Y, and the associated involution* $X \to Y$ *is homotopic to J.*

Definition A.14. Consider the set of all quasi-Fuchsian groups whose *bottom* surface X is *fixed*. This set is called the *Bers slice* B_Y *based on* X. The points in the Teichmüller space are represented by the top surfaces and their relation (via involution) to the *fixed* bottom surface X.

Remark A.15. *The motion through the Bers slice is associated with the motion through various hyperbolic* 3*-manifolds whose bottom boundary component is fixed but whose top varies. As a result of such motion, the top component may eventually degenerate in some way.*

Remark A.16. (a) *The motion through the Bers slice is connected with the motion caused by the pseudo-Anosov surface homeomorphisms (Theorem* 6.20). (b) *The degeneration of the Bers slice is associated with the boundary of the Teichmüller space and formation of all kinds of spikes (cusps) on the top surface.*

Definition A.17. An end of 3-manifold is simply degenerate if it is topologically equivalent (homeomorphic) to the product $S \times R$ where S is the compact Riemann surface of negative Euler characteristic [97].

Definition A.18. 3-manifold is called *geometrically tame* if all of its ends are either geometrically finite (Definition A.3) or simply degenerate (Definition A.17). 3-manifold is *topologically tame* if it is homeomorphic to the interior of compact 3-manifold.

Definition A.19. The thick–thin decomposition of a hyperbolic manifold M [36] can be best understood by analyzing Fig. 6. The thin part of M is made of neigbourhoods of short geodesics and cusps isomorphic to pseudospheres.

Theorem A.20. *The complete hyperbolic manifold M has the finite hyperbolic volume if and only if the thick part,* $M_{\geq\varepsilon}$*, is compact for all* $\varepsilon > 0$.

Proof. Please, consult Ref. 36. □

Remark A.21. *This theorem is central for the whole chain of arguments which provide justification of knots/links existence in* $2+1$ *gravity. Indeed, guided by this theorem, it is possible now to prove the following theorem.*

Theorem A.22. *Let* $M = H^3/\Gamma$ *be infinite volume topologically tame hyperbolic 3-manifold. Then, if* $\hat{\Gamma}$ *is finitely generated subgroup of* Γ, *either*
(a) $\hat{M} = H^3/\hat{\Gamma}$ *is geometrically finite, or*
(b) *the thin part* $M_{\leq\varepsilon}$ *has a geometrically infinite end* $\hat{E}$ *such that the local isometry* $p : \hat{M} \to M$ *is finite one-to-one on some neigbourhood* $\hat{U}$ *of* $\hat{E}$.

Proof. Please, consult Ref. [98] and take into account Fig. 6 and Theorem A.20. □

Remark A.23. *Although Theorem* A.22 *provides the desired result, it does not contain the constructive prescription as to how to obtain* $\hat{M}$ *(that is how to find* $\hat{\Gamma}$ *and to construct the quotient). The most spectacular results of Ref. [70] address just this problem. They are the following:*
(a) *the manifold* $\hat{M}$ *is homeomorphic to the hyperbolic* 3*-manifold which fibers over the circle;*
(b) *the motion through the Bers slice which produces* $\hat{M}$ *is controlled by the functional equation (similar to the Feigenbaum–Cvitanovich equation for the chaotic maps* [99]*). The existence of* $\hat{M}$ *depends on the existence of the fixed point for this equation.*

Remark A.24. *In view of the Feigenbaum–Cvitanovich universal equation for maps of the interval, it is reasonable to expect some sort of universality in the knot /link structures (types) for* $2+1$ *gravity. This universality should be associated with some special role played by fibered knots and links.*

Remark A.25. *As is well known* [100] *the Alexander polynomial* $\Delta(t)$ *for fibered knots/links is monic. That is the first (and the last) nonzero coefficients of* $\Delta(t)$ *are* ± 1. *Recently, it had been shown* [101] *that such monic polynomial could be obtained with help of Seberg–Witten invariants for a special class of 4-manifolds. These are, essentially, Kaluza–Klein-type products of* 3*-manifolds and* S^1. *Thus, description of* $2+1$ *gravity could be also linked to the description of the associated 4-manifolds. Incidentally, the description of* 3*-manifolds in terms of the associated classes of 4-manifolds was originally proposed by Braam* [102].

References

[1] A. Kholodenko, Use of meanders and train tracks for description of defects and textures in liquid crystals and 2 + 1 gravity, J. Geom. Phys. 33 (2000) 23–58.
[2] J. Hubbard, H. Masur, Quadratic differentials and foliations, Acta Math. 142 (1979) 221–274.
[3] V. Poenaru, Some aspects of the theory of defects in ordered media and gauge fields related to foliations, Comm. Math. Phys. 80 (1981) 127–136.
[4] R. Langevin, Foliations, energies and liquid crystals, Asterisque 107–108 (1983) 201–213.
[5] B. Zweibach, How covariant closed string theory solves the minimal area problem, Comm. Math. Phys. 136 (1991) 83–118.

[6] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. 147 (1992) 1–23.
[7] K. Strebel, Quadratic Differentials, Springer, Berlin, 1984.
[8] F. Gardiner, Teicmüller Theory and Quadratic Differentials, Wiley Interscience, New York, 1987.
[9] S. Deser, R. Jackiw, G. 't Hooft, Three dimensional Einstein gravity: dynamics of flat space, Ann. Phys. 152 (1984) 220–235.
[10] M. Troyanov, Les surfaces euclidiennes a singularities coniques, L'Enseignement Math. 32 (1986) 79–94.
[11] I. Rivin, Euclidean structures on simplicial surfaces and hyperbolic volume, Ann. Math. 139 (1994) 553–580.
[12] H. Hopf, Differential Geometry in the Large, LNM No. 1000, Springer, Berlin, 1989.
[13] H. Masur, J. Smillie, Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms, Comm. Math. Helvetici 68 (1993) 289–307.
[14] L. Takhtadjian, Topics in Quantum Geometry of Riemann Surfaces: Two Dimensional Quantum Gravity, hep-th /9409088.
[15] V. Moncrief, Reduction of the Einstein equations in 2 + 1 dimensions to a Hamiltonian system over Teichmüller space, J. Math. Phys. 30 (1989) 2907–2914.
[16] A. Kholodenko, T. Vilgis, Some geometrical and topological problems in polymer physics Phys. Rep. 298 (1998) 251–372.
[17] A. Kholodenko, Statistical mechanics of the deformable droplets on flat surfaces J. Math. Phys. 37 (1996) 1287–1313.
[18] D. Mitrinovich, Analytic Inequalities, Springer, Berlin, 1970.
[19] T. Rado, Length and Area, Am. Math. Soc. Colloq. Publ. 30, Am. Math. Soc., Providence, RI, 1948.
[20] R. Osserman, The isoperimetric inequality, Bull. Am. Math. Soc. 84 (1978) 1182–1238.
[21] L. Ahlfors, Lectures on Quasiconformal Mappings, van Nostrand, NJ, Princeton, 1966.
[22] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. 1, Academic press, New York, 1972.
[23] S. Chandrasekhar, Liquid Crystals, Cambridge University Press, Cambridge, 1992.
[24] G. Levitt, Foliations and laminations on hyperbolic surfaces, Topology 22 (1983) 119–135.
[25] K. Strebel, On quadratic differentials with closed trajectories and second order poles, J. Analyse Math. 19 (1983) 373–382.
[26] K. Strebel, On quadratic differentials with closed trajectories on open Riemann surfaces, Ann. Acad. Sci. Fenn. A 2 (1976) 533–551.
[27] P. Buser, Geometry and Spectra of Compact Riemann Surfaces, Birkhäuser, Boston, MA, 1992.
[28] A. Marden, Geometric relations between homeomorphic Riemann surfaces, Bull. Am. Math. Soc. (New Series) 3 (1980) 1001–1017.
[29] W. Abikoff, The Real Analytic Theory of Teichmüller Space, Springer, Berlin, New York, 1980.
[30] S. Chern in Analysis Et Cetera, Academic Press, New York, 1980.
[31] W. Fulton, Algebraic topology. A first Course, Springer, Berlin, 1995.
[32] A. Casson, S. Bleiler, Automorphisms of Surfaces After Nielsen and Thurston, Cambridge University Press, Cambridge, 1993.
[33] H. Poincare, Collected Works, Nauka, Moscow, 1971.
[34] A. Fathi, F. Laudenbach, V. Poenaru, Travaux de Thurston sur les diffeomorphismes des surfaces et l'espace de Teichmüller, Asterisque 66–67 (1979).
[35] J.-P. Otal, Le theoreme d'hyperbolization pour les varietes fibres de dimension 3, Asterisque 235 (1996).
[36] W. Thurston, Three-Dimensional Geometry and Topology, vol. 1, Princeton University Press, Princeton, NJ, 1997.
[37] P.-G. de Gennes, The Physics of Liquid Crystals, Clarendon Press, Oxford, 1979.
[38] S. Singh, Curvature elasticity in liquid crystals, Phys. Rep. 277 (1996) 283–386.
[39] A. Polyakov, Gauge Fields and Strings, Harwood Academic, New York, 1987.
[40] H. Bresis, J.-M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (1986) 649–705.
[41] H. Bresis, in: Topics in Calculus of Variations, LNM No. 1365, Springer, Berlin, 1988.
[42] F. Bethuel, H. Bresis, F. Helein, Ginzburg-Landau Vortices, Birkhäuser, Boston, MA, 1994.
[43] P. Chaikin, T. Lubensky, Principles of Condensed Matter Physics, Cambridge University Press, Cambridge, 1995.

[44] R. Courant, Geometrisce Function Theorie, Springer, Berlin, 1964.
[45] A. Marden, K. Strebel, The heights theorem for quadratic differentials on Riemann surfaces, Acta Math. 153 (1984) 153–211.
[46] J. Stillwell, Geometry of Surfaces, Springer, Berlin, 1992.
[47] B. Halperin, D. Nelson, Theory of two dimensional melting, Phys. Rev. Lett. 41 (1978) 121.
[48] J.-B. Bost, in: From Number Theory to Physics, Springer, Berlin, 1992.
[49] L. Hugston, K. Tod, An Introduction to General Relativity, Cambridge University Press, Cambridge, 1990.
[50] R. Wald, General Relativity, University of Chicago Press, Chicago, 1984.
[51] A. Petrov, Einstein Spaces, Pergamon, Press, London, 1969.
[52] B. Dubrovin, A. Fomenko, S. Novikov, Modern Geometry–Methods and Applications, vol. 1, Springer, Berlin, 1984.
[53] E. Witten, 2 + 1 dimensional gravity as an exactly soluble problem, Nucl. Phys. B 311 (1988) 46–78.
[54] F. Luo, G. Tian, Liouville equation and spherical complex polytopes, Proc. Am. Math. Soc. 116 (1992) 1119–1129.
[55] R. Mc Owen, Point singularities and conformal metrics on Riemann surface, Proc. Am. Math. Soc. 103 (1988) 222–225.
[56] R. Nevanlinna, Analytic Functions, Springer, Berlin, 1965.
[57] J. Jenkins, Univalent Functions and Conformal Mapping, Springer, Berlin, 1965.
[58] L. Bers, Quasiconformal mapping with applications to differential equations, function theory and topology, Bull. Am. Math. Soc. 83 (1977) 1083–1100.
[59] O. Lehto, Univalent Functions and Teichmüller Spaces, Springer, Berlin, 1987.
[60] L. Bers, An extremal problem for quasiconformal mapping and a theorem by Thurston, Acta Math. 141 (1978) 73–98.
[61] A. Marden, K. Strebel, Pseudo-Anosov Teichmüller mapping, J. Analyse Math. 46 (1986) 194–220.
[62] L. Slutskin, Classification of lifts of automorphisms of surfaces to the unit disc, Contemporary Math. 152 (1993) 311–340.
[63] J. Hempel, 3-Manifolds, Princeton University Press, Princeton, NJ, 1976.
[64] K. Johannson, Topology and Combinatorics of 3-Manifolds, Springer, Berlin, 1995.
[65] G. Hemion, The Classification of Knots and 3-Dimensional Spaces, Oxford University Press. Oxford, 1992.
[66] W.B.R. Lickorish, A representation of orientable combinatorial 3 manifolds, Ann. Math. 76 (1962) 531–540.
[67] A. Wallace, Modifications and coboundary manifolds, Canadian J. Math. 12 (1960) 503–528.
[68] R. Geroch, Domain of dependence, J. Math. Phys. 11 (1970) 437–449.
[69] S. Hawking, G. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge, 1973.
[70] C. McMullen, Renormalization and 3-Manifolds Which Fiber Over the Circle, Princeton University Press, Princeton, NJ, 1996.
[71] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math. 130 (1972) 89–126.
[72] E. D'Hoker, D. Phong, The geometry of string perturbation theory, Rev. Mod. Phys. 60 (1988) 917–1066.
[73] J. Baez (Ed.), Knots and Quantum Gravity, Clarendon Press, Oxford, 1994.
[74] J. Harer, How to construct all fibered knots and links, Topology 21 (1982) 263–280.
[75] G.'t Hooft, Causality in 2 + 1 dimensional gravity, Class. Quantum Grav. 9 (1992) 1335–1348.
[76] G.'t Hooft, Evolution of gravitating point particles in 2 + 1 dimensions, Class. Quantum Grav. 10 (1993) 1023–1038.
[77] G.'t Hooft, Canonical quantization of gravitating point particles in 2 + 1 dimensions, Class. Quantum Grav. 10 (1993) 1653–1664.
[78] G.'t Hooft, Quantization of point particles in 2 + 1 dimensions and spacetime discretness, Class. Quantum Grav. 13 (1996) 1023–1039.
[79] R. Franzosi, E. Guadagnini, Topology and classical geometry in 2 + 1 gravity, Class. Quantum Grav. 13 (1996) 433–460.
[80] A. Connes, D. Sullivan, N. Teleman, Quasiconformal mappings, operators on Hilbert space and local formulas for characteristic classes, Topology 33 (1994) 663–684.
[81] W. Veech, The Teichmüller geodesic flow, Ann. Math. 124 (1986) 441–530

[82] S. Kerkhoff, H. Masur, J. Smillie, Ergodicity of billiard flows and quadratic differentials, Ann. Math. 124 (1986) 293–311.
[83] Y. Vorobets, Plane structures and billiards in rational polygons: the Veech alternative, Russian Math. Surveys 51 (1996) 779–817.
[84] H. Masur, Closed trajectories for quadratic differentials with an application to billiards, Duke Math. J. 53 (1986) 307–314.
[85] H. Masur, Interval exchange transformations and measured foliations, Ann. Math. 115 (1982) 169–200.
[86] Special Issue on Mesoscopic Systems, J. Math. Phys. 37 (10) (1996).
[87] A. Lobkovsky, T. Witten, Properties of ridges in elastic membranes, Phys. Rev. E 55 (1977) 1577–1599.
[88] A. Lobkovsky, S. Gentges, H. Li, D. Morse, T. Witten, Scaling properties of stretching ridges in a crumpled elastic sheet, Science 270 (1995) 1482–1485.
[89] K. Leung, J. Andersen, Phase transition in aspring-block model of surface fracture, Europhys. Lett. 38 (1997) 589–594.
[90] O. Huseby, J.-F. Thovert, P. Adler, Geometry and topology of fracture systems, J. Phys. A 30 (1997) 1415–1444.
[91] A. Buchel, J. Setna, Statistical mechanics of cracks: fluctuations, breakdown, and asymptotics of elastic theory, Phys. Rev. E 55 (1997) 7669–7690.
[92] W.B.R. Lickorish, Invariants for 3-manifolds from the combinatorics of the Jones polynomial, Pac. J. Math. 149 (1991) 337–347.
[93] W.B.R. Lickorish, Three-manifolds and the Temperley–Lieb algebra, Math. Ann. 290 (1991) 657–670.
[94] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1988) 351–399.
[95] B. Westbury, The representation theory for the Temperley–Lieb algebra, Math. Zeitschrift 219 (1995) 539–565.
[96] A. Marden, Geometric relations between homeomorphic Riemann surfaces, Bull. Am. Math. Soc. (New Series) 3 (1980) 1001–1017.
[97] Y. Minsky, Teichmüller geodesics and ends of hyperbolic 3-manifolds, Topology 32 (1993) 625–647.
[98] R. Canary, A covering theorem for hyperbolic 3-manifolds and its applications, Topology 35 (1996) 751–778.
[99] P. Cvitanovich, Universality in Chaos, Adam Hilger, Bristol, 1984.
[100] D. Rolfsen, Knots and Links, Publish or Perish, Houston, 1990.
[101] R. Fintushel, R. Stern, Knots,links and 4-manifolds, Inv. Math. 134 (1998) 363–400.
[102] P. Braam, A. Kaluza-Klein approach to hyperbolic 3-manifolds, L'Enseign. Math. 34 (1988) 275–311.
[103] A. Kholoclenko, Satistical mechanics of the deformable droplets on Riemann surfaces: applications to reptation and related problems, J. Math. Phys. 37 (1996) 1314–1335.